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Fixed Point Results of a New Family of Contractions in Metric Space Endowed with Graph †

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Abstract: One of the applicable concepts in metric fixed point theory is the notion of hybrid functional equations. In the same vein, the role of graphs in computational sciences and nonlinear functional analysis is currently well known. However, as duly revealed from the available literature, we understand that hybrid fixed point notions in metric space endowed with graph have not been well considered. In this note, therefore, a general family of contractive inequality, namely admissible hybrid $(H$-α-$\phi)$-contraction is proposed in metric space equipped with a graph and new criteria for which the mapping is a Picard operator are examined. The significance of this type of contraction is connected with the possibility that its inequality can be particularized in more than one way, depending on the provided constants. A relevant example is designed to support the assumptions of our obtained notions and to show how they are different from the known ones.

Keywords: Metric space; fixed point; hybrid contraction; connected graph; Picard operator

MSC: 47H10; 54H25; 46T99; 46N40; 05C40

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1. Introduction and Preliminaries

The Banach contraction principle in metric space has laid the foundation of modern metric fixed point theory. The importance of fixed point results via this principle runs over several fields of sciences and engineering. Examiners in this area have investigated a lot of novel ideas in metric space and have presented more than a handful of significant results. Lately, Karapınar and Fulga [1] studied a new form of contractive inequality under the name admissible hybrid contraction.

Here and below, (Φ, ϖ) is a metric space, $\Gamma : \Phi \longrightarrow \Phi$ is a self-mapping of Φ and $\mathbb{N}$ is the set of natural numbers.

Following Petruşel and Rus [2], Γ is termed a Picard operator if Γ has a unique fixed point r^* and for all $r \in \Phi$, $\lim_{p\to\infty} \Gamma^p r = r^*$.

Jachymski [3] introduced the notion of graphic contraction in metric space. Accordingly, let Δ be the diagonal of the Cartesian product $\Phi \times \Phi$. Imagine a directed graph H where the set of its edges $E(H)$ contains all loops and the set of its vertices $V(H)$ coincides with Φ. As it is assumed, H does not have any parallel edges and can therefore, be represented by the pair $(V(H), E(H))$. In addition, H can be thought of as a weighted graph by giving each edge the distance between its vertices (see ([4], p. 376)). The undirected graph $\tilde{H}$ is made from H by ignoring edge direction or, more easily, by treating H as a directed graph whose set of edges is symmetric.

For any two vertices r and s in a graph H, a path in H from r to s of length $N \in \mathbb{N}$ is a sequence $\{r_i\}_{i=0}^{N}$ of $N+1$ vertices in the sense that $r_0 = r$, $r_N = s$ and $(r_{p-1}, r_p) \in E(H)$

for all $i = 1, 2, ..., N$. A graph H is connected if a path exists between any two vertices and weakly connected if $\tilde{H}$ is connected.

Multiple writers ([5–7], for example,) have established fixed point results involving Lipschitzian-type mappings in metric spaces equipped with graph.

Definition 1. *[8] For a given function $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$, Γ is termed α-orbital admissible if for all $r \in \Phi$,*

$$\alpha(r, \Gamma r) \geq 1 \Rightarrow \alpha(\Gamma r, \Gamma^2 r) \geq 1.$$

Definition 2. *[8] For a given function $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$, Γ is termed triangular α-orbital admissible if for all $r \in \Phi$, Γ is α-orbital admissible and*

$$\alpha(r, s) \geq 1 \text{ and } \alpha(s, \Gamma s) \geq 1 \Rightarrow \alpha(r, \Gamma s) \geq 1.$$

Definition 3. *[5] The mapping Γ is said to be orbitally continuous if for all $r \in \Phi$ and any sequence $\{k_p\}_{p\in\mathbb{N}}$, $\Gamma^{k_p} r \longrightarrow s \in \Phi$ implies that $\Gamma(\Gamma^{k_p} r) \longrightarrow \Gamma s$ as $p \to \infty$.*

Definition 4. *[5] The mapping Γ is said to be orbitally H-continuous if for all $r \in \Phi$ and a sequence $\{r_p\}_{p\in\mathbb{N}}$, $r_p \longrightarrow r$ and $(r_p, r_{p+1}) \in E(H)$ imply that $\Gamma r_p \longrightarrow \Gamma r$ as $p \to \infty$.*

As duly revealed from the available literature, we understand that hybrid fixed point concepts in metric space endowed with graph have not been well considered. Whence, invited by the ideas in [1,3,5], we initiate an idea of admissible hybrid (H-α-ϕ)-contraction in metric space equipped with graph and investigate the conditions for which this new contraction is a Picard operator. A substantial example is constructed to demonstrate that our obtained result is valid and distinct from the existing results in the literature.

2. Main Results

We now examine the idea of admissible hybrid (H-α-ϕ)-contraction in metric space endowed with a graph H.

Definition 5. *For any (Φ, ϖ) endowed with a graph H, Γ is termed an admissible hybrid (H-α-ϕ)-contraction if:*

(i) $\forall r, s \in \Phi\big((r, s) \in E(H) \Rightarrow (\Gamma r, \Gamma s) \in E(H)\big)$;

(ii) *we can find $\phi \in \Psi$ and $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ such that*

$$\alpha(r, s)\varpi(\Gamma r, \Gamma s) \leq \phi(\mathcal{H}_{\mathcal{A}}(r, s)) \tag{2.1}$$

for all $(r, s) \in E(H)$, where

$$\mathcal{H}_{\mathcal{A}}(r,s) = \begin{cases} \Big[\lambda_1 \varpi(r,s)^q + \lambda_2 \varpi(r,\Gamma r)^q + \lambda_3 \varpi(s,\Gamma s)^q \\ + \lambda_4 \Big(\frac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\Big)^q + \lambda_5 \Big(\frac{\varpi(s,\Gamma r)(1+\varpi(r,\Gamma s))}{1+\varpi(r,s)}\Big)^q\Big]^{\frac{1}{q}} \\ \qquad\qquad \text{for} \quad q > 0; \\ \\ [\varpi(r,s)]^{\lambda_1} \cdot [\varpi(r,\Gamma r)]^{\lambda_2} \cdot [\varpi(s,\Gamma s)]^{\lambda_3} \cdot \Big[\frac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\Big]^{\lambda_4} \\ \cdot \Big[\frac{\varpi(r,\Gamma s)+\varpi(s,\Gamma r)}{2}\Big]^{\lambda_5} \quad \text{for} \quad q = 0,\ r \neq s,\ r \neq \Gamma r, \end{cases}$$

$\lambda_i \geq 0$ with $i = 1, 2, ..., 5$, $\sum_{i=1}^{5} \lambda_i = 1$ and Ψ is the set of all functions $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying:

(ϕ_i) ϕ is monotone increasing;

(ϕ_{ii}) *the series* $\sum_{p=0}^{\infty} \phi^p(t)$ *is convergent for all* $t > 0$.

The following is our main result.

Theorem 1. *On a complete* (Φ, ϖ) *endowed with a graph* H *and an admissible hybrid* $(H\text{-}\alpha\text{-}\phi)$*-contraction* Γ, *if we assume further that:*

(i) Γ *is triangular* α*-orbital admissible;*
(ii) *we can find* $r_0 \in \Phi$ *such that* $\alpha(r_0, \Gamma r_0) \geq 1$;
(iii) H *is weakly connected;*
(iv) *for any sequence* $\{r_p\}_{p\in\mathbb{N}}$ *in* Φ *with* $\varpi(r_p, r_{p+1}) \longrightarrow 0$, *we can find* $k, p_0 \in \mathbb{N}$ *such that* $(r_{kp}, r_{km}) \in E(H)$ *for all* $p, m \in \mathbb{N}$, $p, m \geq p_0$;
$(v)_a$ Γ *is orbitally continuous or;*
$(v)_b$ Γ *is orbitally* H*-continuous and we can find a subsequence* $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}}$ *of* $\{\Gamma^p r_0\}_{p\in\mathbb{N}}$ *such that* $(\Gamma^{p_k} r_0, r^*) \in E(H)$ *for each* $k \in \mathbb{N}$.

Then Γ *is a Picard operator.*

Example 1. *Let* $\Phi = \{1, 2, 3, 4, 5, 6\}$ *be endowed with the metric* $\varpi : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ *defined by*

$$\varpi(r, s) = |r - s|, \quad \forall r, s \in \Phi.$$

Then (Φ, ϖ) *is a complete metric space.*
Consider a mapping $\Gamma : \Phi \longrightarrow \Phi$ *given by*

$$\Gamma r = \begin{cases} \frac{r}{2}, & \text{if } r \in \{2, 4, 6\}; \\ 1, & \text{if } r \in \{1, 3, 5\} \end{cases}$$

for all $r \in \Phi$ *and* $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ *by*

$$\alpha(r, s) = \begin{cases} 2, & \text{if } r, s \in \{4, 5\}; \\ 1, & \text{otherwise.} \end{cases}$$

Consider the symmetric graph $\tilde{H}$ *defined by* $V(\tilde{H}) = \Phi$ *and*

$$E(\tilde{H}) = \{(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,5), (4,6), (5,6)\} \cup \Delta.$$

Then it is clear that Γ *preserves edges,* Γ *is triangular* α*-orbital admissible and* H *is weakly connected. To see that* Γ *is an admissible hybrid* $(H\text{-}\alpha\text{-}\phi)$*-contraction, let* $\phi(t) = \frac{9t}{10}$ *for all* $t \geq 0$, $\lambda_1 = \lambda_5 = \frac{1}{10}$, $\lambda_2 = \lambda_4 = \frac{2}{5}$ *and* $\lambda_3 = 0$ *for* $q = 0, 2$. *We then consider the following cases:*

Case 1: $r = s$, $r, s \in \{2, 4, 6\}$;
Case 2: $r \neq s$, $r, s \in \{2, 4, 6\}$;
Case 3: $r = s$, $r, s \in \{1, 3, 5\}$;
Case 4: $r \neq s$, $r, s \in \{1, 3, 5\}$;
Case 5: $r \in \{2, 4, 6\}$ *and* $s \in \{1, 3, 5\}$;
Case 6: $r \in \{1, 3, 5\}$ *and* $s \in \{2, 4, 6\}$.

We demonstrate using the following Tables 1 and 2 that inequality (2.1) is satisfied for each of the above cases.

In the following Figures 1 and 2, we present the symmetric graph $\tilde{H}$ *defined in Example 1 and illustrate the validity of contractive inequality (2.1) using Example 1.*

Therefore, all the hypotheses of Theorem 1 are satisfied, Γ *has a unique fixed point,* $r = 1$ *and* $\lim_{p\to\infty} \Gamma^p r = 1$ *for all* $r \in \Phi$. *Consequently,* Γ *is a Picard operator.*

Table 1. Table of values for Cases 1–4.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)), q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)), q=2$
Case 1	2	2	0	-	1.32272
	4	4	0	-	3.88566
	6	6	0	-	7.71274
Case 2	4	6	1	2.16471	2.01067
	6	4	1	2.42870	1.93075
Case 3	1	1	0	-	0
	3	3	0	-	3.88566
	5	5	0	-	12.80912
Case 4	1	3	0	-	0.46086
	3	1	0	-	0.74215
	3	5	0	2.47339	2.65156
	5	3	0	3.03412	2.57021

Table 2. Table of values for Cases 5 and 6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)), q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)), q=2$
Case 5	2	1	0	-	0.37107
	2	3	0	1.23669	1.32578
	2	5	0	1.45264	1.35
	4	3	1	1.97517	1.94074
	4	5	2	2.71410	3.88670
	6	5	2	3.63692	4.98261
Case 6	1	2	0	-	0.33068
	3	2	0	1.45445	1.19906
	3	4	1	1.97517	2.04242
	5	2	0	2.09573	1.66712
	5	4	2	3.32940	3.61907
	5	6	2	3.97647	4.98627

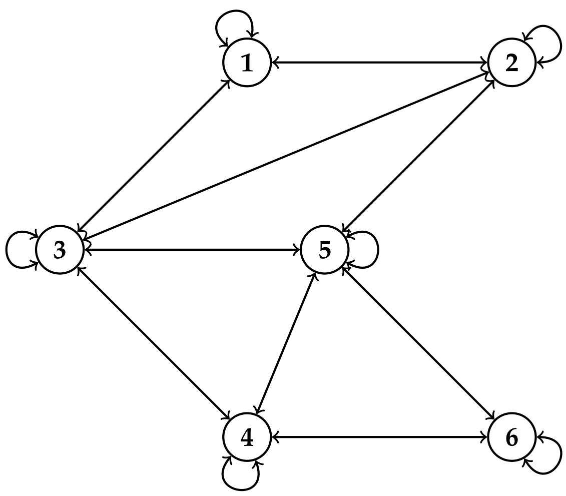

Figure 1. Symmetric graph $\tilde{H}$ defined in Example 1

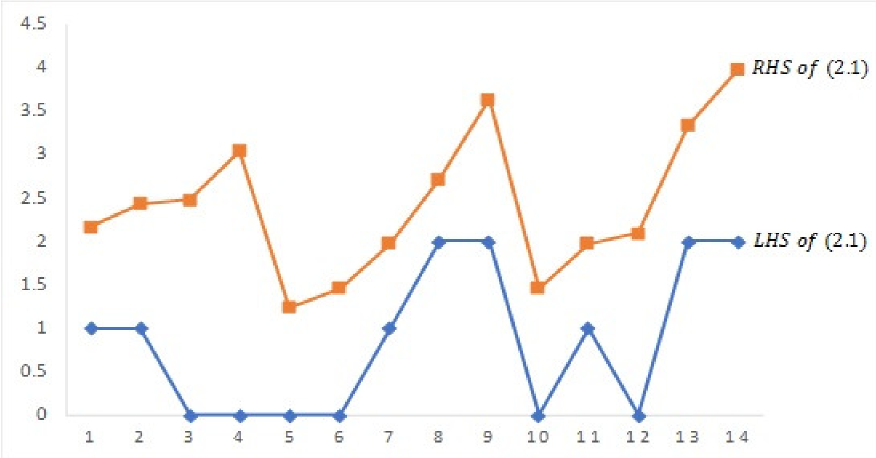

Figure 2. Illustration of contractive inequality (2.1) for $q = 0$.

References

1. Karapınar, E.; Fulga, A. An Admissible Hybrid Contraction With An Ulam Type Stability. *Demonstr. Math.* **2019**, *52*, 428–436.
2. Petruşel, A.; Rus, I. Fixed Point Theorems in Ordered $\mathcal{L}$-spaces. *Proc. Am. Math. Soc.* **2006**, *134*, 411–418.
3. Jachymski, J. The Contraction Principle for Mappings on a Metric Space with a Graph. *Proc. Am. Math. Soc.* **2008**, *1*, 1359–1373.
4. Johnsonbaugh, R. *Discrete Mathematics*, 8th ed; Pearson Education Inc.: London, UK, 2018.
5. Bojor, F. Fixed Point of φ-contraction in Metric Spaces with a Graph. *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **2010**, *37*, 85–92.
6. Bojor, F. Fixed Point Theorems for Reich Type Contractions on Metric Spaces with a Graph. *J. Nonlinear Anal.* **2012**, *75*, 3895–3901.
7. Bojor, F. Fixed Points of Kannan Mappings in Metric Spaces Endowed with a Graph. *Math. J. Ovidius Univ. Constantza* **2012**, *20*, 31–40.
8. Popescu, O. Some New Fixed Point Theorems for α-Geraghty Contraction Type Maps in Metric Spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 190.