Fixed Point Results of a New Family of Contractions in Metric Space Endowed with Graph

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Outline

Abstract

One of the applicable concepts in metric fixed point theory is the notion of hybrid functional equations. In the same vein, the role of graphs in computational sciences and nonlinear functional analysis is currently well known. However, as duly revealed from the available literature, we understand that hybrid fixed point notions in metric space endowed with graph have not been well considered. In this note, therefore, a general family of contractive inequality, namely admissible hybrid (H-α-ϕ)-contraction is proposed in metric space equipped with a graph and new criteria for which the mapping is a Picard operator are examined. The significance of this type of contraction is connected with the possibility that its inequality can be particularized in more than one way, depending on the provided constants. A relevant example is designed to support the assumptions of our obtained notions and to show how they are different from the known ones.

Introduction and Preliminaries

The Banach contraction principle in metric space has laid the foundation of modern metric fixed point theory. The importance of fixed point results via this principle runs over several fields of sciences and engineering. Examiners in this area have investigated a lot of novel ideas in metric space and have presented more than a handful of significant results. Lately, Karapınar and Fulga (2019a) studied a new form of contractive inequality under the name admissible hybrid contraction.

Introduction and Preliminaries (Cont'd)

Here and below, (Φ, ϖ) is a metric space, $\Gamma : \Phi \longrightarrow \Phi$ is a self-mapping of Φ and $\mathbb{N}$ is the set of natural numbers.
Following Petruşel and Rus (2006), Γ is termed a Picard operator if Γ has a unique fixed point r^* and for all $r \in \Phi$, $\lim_{p\to\infty} \Gamma^p r = r^*$.

Introduction and Preliminaries (Cont'd)

Jachymski (2008) introduced the notion of graphic contraction in metric space. Accordingly, let Δ be the diagonal of the Cartesian product $\Phi \times \Phi$. Imagine a directed graph H where the set of its edges $E(H)$ contains all loops and the set of its vertices $V(H)$ coincides with Φ. As it is assumed, H does not have any parallel edges and can therefore, be represented by the pair $(V(H), E(H))$. In addition, H can be thought of as a weighted graph by giving each edge the distance between its vertices (see [Johnsonbaugh (2018), p. 376]). The undirected graph $\tilde{H}$ is made from H by ignoring edge direction or, more easily, by treating H as a directed graph whose set of edges is symmetric.

Introduction and Preliminaries (Cont'd)

For any two vertices r and s in a graph H, a path in H from r to s of length $N \in \mathbb{N}$ is a sequence $\{r_i\}_{i=0}^{N}$ of $N+1$ vertices in the sense that $r_0 = r$, $r_N = s$ and $(r_{p-1}, r_p) \in E(H)$ for all $i = 1, 2, \ldots, N$. A graph H is connected if a path exists between any two vertices and weakly connected if $\tilde{H}$ is connected. Multiple writers (Bojor (2010) and Bojor (2012b), for example,) have established fixed point results involving Lipschitzian-type mappings in metric spaces equipped with graph.

Introduction and Preliminaries (Cont'd)

Definition 1

(Popescu 2014) For a given function $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$, Γ is termed α-orbital admissible if for all $r \in \Phi$,

$$\alpha(r, \Gamma r) \geq 1 \Rightarrow \alpha(\Gamma r, \Gamma^2 r) \geq 1.$$

Definition 2

(Popescu 2014) For a given function $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$, Γ is termed triangular α-orbital admissible if for all $r \in \Phi$, Γ is α-orbital admissible and

$$\alpha(r, s) \geq 1 \text{ and } \alpha(s, \Gamma s) \geq 1 \Rightarrow \alpha(r, \Gamma s) \geq 1.$$

Introduction and Preliminaries (Cont'd)

Definition 3

(Bojor 2010) The mapping Γ is said to be orbitally continuous if for all $r \in \Phi$ and any sequence $\{k_p\}_{p\in\mathbb{N}}$, $\Gamma^{k_p} r \longrightarrow s \in \Phi$ implies that $\Gamma(\Gamma^{k_p} r) \longrightarrow \Gamma s$ as $p \to \infty$.

Definition 4

(Bojor 2010) The mapping Γ is said to be orbitally H-continuous if for all $r \in \Phi$ and a sequence $\{r_p\}_{p\in\mathbb{N}}$, $r_p \longrightarrow r$ and $(r_p, r_{p+1}) \in E(H)$ imply that $\Gamma r_p \longrightarrow \Gamma r$ as $p \to \infty$.

Introduction and Preliminaries (Cont'd)

As duly revealed from the available literature, we understand that hybrid fixed point concepts in metric space endowed with graph have not been well considered. Whence, invited by the ideas in Jachymski (2008), Karapınar and Fulga (2019a) and Bojor (2010), we initiate an idea of admissible hybrid (H-α-ϕ)-contraction in metric space equipped with graph and investigate the conditions for which this new contraction is a Picard operator. A substantial example is constructed to demonstrate that our obtained result is valid and distinct from the existing results in the literature.

Main Results

- **Admissible hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction**
  We now examine the idea of admissible hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction in metric space endowed with a graph H.

Definition 5

For any (Φ, ϖ) endowed with a graph H, Γ is termed an admissible hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction if:

(i) $\forall r, s \in \Phi\big((r, s) \in E(H) \Rightarrow (\Gamma r, \Gamma s) \in E(H)\big)$;

(ii) we can find $\phi \in \Psi$ and $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ such that

$$\alpha(r, s)\varpi(\Gamma r, \Gamma s) \leq \phi(\mathcal{H}_{\mathcal{A}}(r, s)) \tag{1}$$

for all $(r, s) \in E(H)$, where

Main Results (Cont'd)

- **Admissible hybrid (H-α-ϕ)-contraction**

$$\mathcal{H}_{\mathcal{A}}(r,s) = \begin{cases} \Big[\lambda_1 \varpi(r,s)^q + \lambda_2 \varpi(r,\Gamma r)^q + \lambda_3 \varpi(s,\Gamma s)^q \\ +\lambda_4 \left(\frac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right)^q + \lambda_5 \left(\frac{\varpi(s,\Gamma r)(1+\varpi(r,\Gamma s))}{1+\varpi(r,s)}\right)^q \Big]^{\frac{1}{q}} \\ \qquad for \quad q > 0; \\ \\ [\varpi(r,s)]^{\lambda_1} \cdot [\varpi(r,\Gamma r)]^{\lambda_2} \cdot [\varpi(s,\Gamma s)]^{\lambda_3} \cdot \left[\frac{\varpi(s,\Gamma s)(1+\varpi(r,\Gamma r))}{1+\varpi(r,s)}\right]^{\lambda_4} \\ \cdot \left[\frac{\varpi(r,\Gamma s)+\varpi(s,\Gamma r)}{2}\right]^{\lambda_5} \quad for \quad q = 0,\ r \neq s,\ r \neq \Gamma r, \end{cases}$$

$\lambda_i \geq 0$ with $i = 1, 2, ..., 5$, $\sum_{i=1}^{5} \lambda_i = 1$ and Ψ is the set of all functions $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying:

(ϕ_i) ϕ is monotone increasing;

(ϕ_{ii}) the series $\sum_{p=0}^{\infty} \phi^p(t)$ is convergent for all $t > 0$.

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

Theorem 6

On a complete (Φ, ϖ) endowed with a graph H and an admissible hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction Γ, if we assume further that:

(i) Γ *is triangular* α*-orbital admissible;*

(ii) *we can find* $r_0 \in \Phi$ *such that* $\alpha(r_0, \Gamma r_0) \geq 1$*;*

(iii) H *is weakly connected;*

(iv) *for any sequence* $\{r_p\}_{p\in\mathbb{N}}$ *in* Φ *with* $\varpi(r_p, r_{p+1}) \longrightarrow 0$*, we can find* $k, p_0 \in \mathbb{N}$ *such that* $(r_{kp}, r_{km}) \in E(H)$ *for all* $p, m \in \mathbb{N}$*,* $p, m \geq p_0$*;*

$(v)_a$ Γ *is orbitally continuous or;*

$(v)_b$ Γ *is orbitally* H*-continuous and we can find a subsequence* $\{\Gamma^{p_k} r_0\}_{k\in\mathbb{N}}$ *of* $\{\Gamma^p r_0\}_{p\in\mathbb{N}}$ *such that* $(\Gamma^{p_k} r_0, r^*) \in E(H)$ *for each* $k \in \mathbb{N}$*.*

Then Γ *is a Picard operator.*

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

Example 7

Let $\Phi = \{1, 2, 3, 4, 5, 6\}$ be endowed with the metric $\varpi : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ defined by

$$\varpi(r, s) = |r - s|, \quad \forall r, s \in \Phi.$$

Then (Φ, ϖ) is a complete metric space. Consider a mapping $\Gamma : \Phi \longrightarrow \Phi$ given by

$$\Gamma r = \begin{cases} \frac{r}{2}, & \text{if } r \in \{2, 4, 6\}; \\ 1, & \text{if } r \in \{1, 3, 5\} \end{cases}$$

for all $r \in \Phi$ and

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

$\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}^+$ by

$$\alpha(r, s) = \begin{cases} 2, & \text{if } r, s \in \{4, 5\}; \\ 1, & \text{otherwise.} \end{cases}$$

Also, consider the symmetric graph $\tilde{H}$ defined by $V(\tilde{H}) = \Phi$ and

$$E(\tilde{H}) = \{(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,5), (4,6), (5,6)\} \cup \Delta.$$

Then it is clear that Γ preserves edges, Γ is triangular α-orbital admissible and H is weakly connected.
To see that Γ is an admissible hybrid (H-α-ϕ)-contraction, let $\phi(t) = \frac{9t}{10}$ for all $t \geq 0$, $\lambda_1 = \lambda_5 = \frac{1}{10}$, $\lambda_2 = \lambda_4 = \frac{2}{5}$ and $\lambda_3 = 0$ for $q = 0, 2$.

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction**
  We then consider the following cases:

Case 1: $r = s$, $r, s \in \{2, 4, 6\}$;
Case 2: $r \neq s$, $r, s \in \{2, 4, 6\}$;
Case 3: $r = s$, $r, s \in \{1, 3, 5\}$;
Case 4: $r \neq s$, $r, s \in \{1, 3, 5\}$;
Case 5: $r \in \{2, 4, 6\}$ and $s \in \{1, 3, 5\}$;
Case 6: $r \in \{1, 3, 5\}$ and $s \in \{2, 4, 6\}$.

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

We demonstrate using the following Tables 1-4, and Figures 1 and 2 that inequality (1) is satisfied for each of the above cases.

Table 1: Table of values for Cases 1-6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)),\ q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s)),\ q=2$
Case 1	2	2	0	-	1.32272
	4	4	0	-	3.88566
	6	6	0	-	7.71274
Case 2	4	6	1	2.16471	2.01067
	6	4	1	2.42870	1.93075

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

Table 2: Table of values for Cases 1-6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=2$
Case 3	1	1	0	-	0
	3	3	0	-	3.88566
	5	5	0	-	12.80912
Case 4	1	3	0	-	0.46086
	3	1	0	-	0.74215
	3	5	0	2.47339	2.65156
	5	3	0	3.03412	2.57021

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

Table 3: Table of values for Cases 1-6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=2$
Case 5	2	1	0	-	0.37107
	2	3	0	1.23669	1.32578
	2	5	0	1.45264	1.35
	4	3	1	1.97517	1.94074
	4	5	2	2.71410	3.88670
	6	5	2	3.63692	4.98261

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

Table 4: Table of values for Cases 1-6.

Cases	r	s	$\alpha(r,s)\varpi(\Gamma r,\Gamma s)$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=0$	$\phi(\mathcal{H}_{\mathcal{A}}(r,s))$, $q=2$
Case 6	1	2	0	-	0.33068
	3	2	0	1.45445	1.19906
	3	4	1	1.97517	2.04242
	5	2	0	2.09573	1.66712
	5	4	2	3.32940	3.61907
	5	6	2	3.97647	4.98627

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid $(H$-α-$\phi)$-contraction**

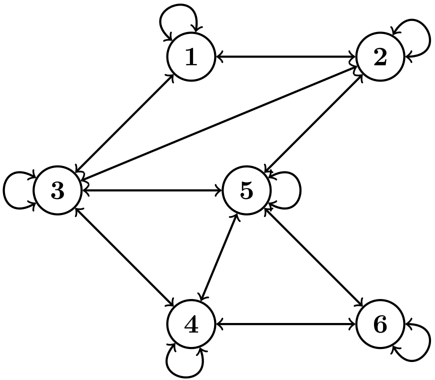

Figure 1: Symmetric graph $\tilde{H}$ defined in Example 7

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid (H-α-ϕ)-contraction**

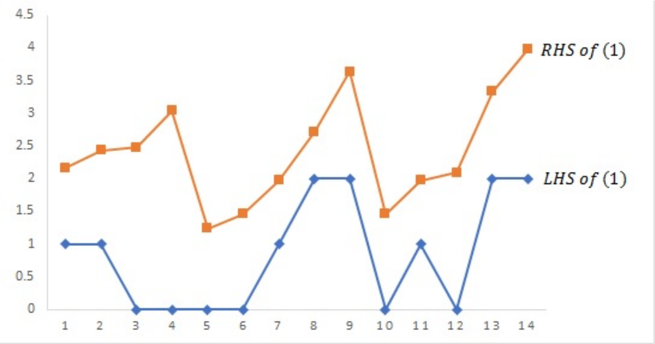

Figure 2: Illustration of contractive inequality (1) for $q = 0$.

Main Results (Cont'd)

- **Fixed Point Results of Admissible Hybrid $(H\text{-}\alpha\text{-}\phi)$-contraction**

Therefore, all the hypotheses of Theorem 6 are satisfied, Γ has a unique fixed point, $r = 1$ and $\lim_{p\to\infty} \Gamma^p r = 1$ for all $r \in \Phi$. Consequently, Γ is a Picard operator.

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THANK YOU!