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A Varadhan estimate for big order differential generators

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Abstract: We give a logaritmic estimate of an elliptic semi-group generated by a big order generator by using the Malliavin Calculus of Bismut type and large deviation estimates.

Keywords: heat kernel; big order generator

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1. Introduction and main results

Let us consider a compact Riemannian manifold M of dimension d endowed with its normalized Riemannian measure dx $(x \in M)$. Associated to it we consider the Laplace-Beltrami operator Δ and the heat semi-group associated to it P_t

$$\partial/(\partial t) P_t f = -\Delta/2 P_t f \tag{1}$$

if f is a smooth function on M.

The heat semi-group is represented by an heat-kernel if $t > 0$

$$P_t f(x) = \int_M p_t(x,y) f(y) dy \tag{2}$$

where $(x,y) \to p_t(x,y)$ is smooth positive.

Associated to the Riemannian structure, we consider the Riemannian distance $(x,y) \to d_R(x,y)$ which is continuous positive. Varadhan's type estimate state that

$$lim_{t\to 0} 2t \log p_t(x,y) = -d_R^2(x,y) \tag{3}$$

For a subelliptic operator, we can consider the associated semi-group. By Hoermander's theorem, there is still associated an heat-kernel. There is the generalization in this case of the Riemannian distance called the Sub-Riemannian distance $d_{s.R}(x,y)$ which is still continuous positive finite. Under some technical conditions by using a mixture of the Malliavin Calculus and large deviation estimates, we have shown in [3]

$$\overline{lim}_{t\to 0} 2t \log p_t(x,y) \leq -d_{s.R}^2(x,y) \tag{4}$$

Our goal is to repeat the strategy of [3] for non-markovian semi group. We consider some vector fields X_i smooth without divergence on the manifold and we consider the operator

$$L = (-1)^k \sum_{i=1}^m X_i^{2k} \tag{5}$$

for some strictly positive integer k. We suppose that at each point x of M, the vector fields spann the tangent space $T_x(M)$. In such a case, the operator is elliptic positive symmetric. By abstract theory [2], it admits a self-adjoint extension and is essentially self-adjoint.

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We consider the heat semi-group associated to it:

$$\partial/(\partial t)P_t^L f = -LP_t^L f \tag{6}$$

if f is a smooth function on M and $P_0^L f = f$. The main difference with the case of the Laplacian is that the semi-group does not preserve the positivity. Classically in analysis, the semi group P_t^L has an heat kernel $p_t^L(x,y)$ which changes of sign [2]. We have shown this result by using the tools of the Malliavin Calculus for non-markovian semi-groups (See [6] for a review).

To L is associated an Hamiltonian H. It is an application on $T^*(M)$, the cotangent bundle of M given by if $\xi \in T_x^*(m)$

$$(x,\xi) \to \sum_{i=1}^{m} <\xi, X_i(x)>^{2k} \tag{7}$$

Due to the hypothesis of ellipticity, we have

$$|H(x,\xi)| \geq C|\xi|^{2k} \tag{8}$$

According to the theory of large deviation, we introduce the Lagrangian associated. It is a function from $T(M)$ the tangent bundle of M into $\mathbb{R}$

$$L(x,p) = \sup_{\xi}(<\xi, p> -H(x,\xi)) \tag{9}$$

where $p \in T_x(M)$. It $t \to \gamma(t)$ is a finite energy curve on M, we define its action

$$S(\gamma) = \int_0^1 L(\gamma(t), d/dt\gamma(t))dt \tag{10}$$

and we put

$$l(x,y) = \inf_{\gamma(0)=x;\gamma(1)=y} S(\gamma) \tag{11}$$

By standard methods, du to the estimate (8), $(x,y) \to l(x,y)$ is continuous.

The goal of this note is to show a Varadhan type estimate for $p_t^L(c,y)$:

Theorem When $t \to 0$, we have uniformly

$$\overline{limt}^{1/(2k-1}\log|p_t^L(x,y)| \leq -l(x,y) \tag{12}$$

This estimate has to be compare with the standard estimates of harmonic analysis (See for instance [1]). We adapt the method of [3] in this non-markovian context. Let us remark that we have already got similar estimates in [4], [5], [6] for right-invariant elliptic operators on compact Lie groups, by mixing tools of the Malliavin Calculus for non-markovian semi-groups and Wentzel-Freidlin estimates for non-Markovian semi groups.

With respect of [1], the variational problem associated to the semi-group appears. The asymptotic is global unlike the standard asymptotics of semi-classical analysis which are instable. The main novelty with respect of the traditional result of stochastic analysis is the new exponent in the asymptotic, which is due to the fact that we consider a big order generator.

In the next part, we prove the theorem by using rough estimates of the heat kernel which are got by the Malliavin Calculus and large deviation estimates on the semi-group.

2. Proof of the theorem

Since L is symmetric

$$\int_M f(x)P_t^L g(x)dx = \int_M g(x)P_t^L f(x)dx \tag{13}$$

such that $p_t^L(x,y) = p_t^L(y,x)$.

Let us recall some results of [6]. In part 4 of [6], we shown by using the intrinsic Malliavin Calculus on the semi-group generated by L that:

$$|p_t^l(x,y)| \leq Ct^{-l} \tag{14}$$

for $t \leq 1$ uniformly in (x,y).

Moreover in part 5 of [6], we have shown if O is an open ball that uniformly in x

$$\overline{lim} t^{1/(2k-1)} \log |P_t^l|(1_O](x) \leq -\inf_{y\in O} l(x,y)) \tag{15}$$

This means in other words that if η is small for small t

$$\int_O |p_t^L(x,y)|dy \leq \exp[(-\inf_{y\in O} l(x,y) + \eta)t^{-(1/(2k-1))}] \tag{16}$$

We have shown in [6] lemma 9 the following result. For all δ, all C, there exists s_δ such that if $s \leq s_\delta$

$$|P_{st}^L|[1_{B(x,\delta)^c}](x) \leq \exp[-Ct^{-1/(2k-1)}] \tag{17}$$

This means that

$$\int_{B(x,\delta)^c} |p_{st}^L|(x,y)dy \leq \exp[-Ct^{-1/(2k-1)}] \tag{18}$$

The two previous estimates are uniform. (14), (15) and (18) will allow to conclude. By the the semi-group property

$$p_t^L(x,y) = \int_M p_{(1-s)t}^L(x,z)p_{st}^L(z,y)dz \tag{19}$$

We deduce that

$$|p_t^L(x,y)| \leq A + B \tag{20}$$

where

$$A = \int_{B(y,\delta)} |p_{(1-s)t}^L||p_{(st)}^L(z,y)|dz \tag{21}$$

and

$$B = \int_{B(y,\delta)^c} |p_{(1-s)t}^L||p_{(st)}^L(z,y)|dz \tag{22}$$

If s is small enough

$$B \leq C((1-s)t)^{-l} \exp[-Ct^{-1/(2k-1)}] \tag{23}$$

If t is small enough

$$A \leq C(st)^{-l} \exp[-(\inf_{z\in B(y,\delta)} l(x,z) + \eta)t^{-(1/(2k-1))})] \tag{24}$$

The conclusion holds if we choose δ very small such that $\inf_{z\in B(y,\delta)} l(x,z)$ is close from $l(x,y)$ because $(x,z) \to l(x,z)$ is continuous.

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