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Proceeding Paper

New Structure of Skew Braces and Their Ideals †

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Abstract: The main reason to write this article is to introduce the definition of a strongly belong element of a brace with some results concerning that, where I acts as an ideal of a brace $(A, +, \circ)$. We named that an element $a \in I$ acts as strongly belong element of I when $a + b$ yields a for all b in I.

Keywords: Brace; skew brace; strongly belong element; left brace

MSC: 12F10; 16T05

1. Introduction

Motivation for writing this paper came from publication of Konovalov et al. [1]. Historically, the Yang-Baxter formula appeared in 1944 as one of the earliest of statistical mechanics. The Yang-Baxter formula has been calculated extensively, and its applications can be established in classical statistical mechanics, quantum groups and other fields.There are several authors investigated the solution of the Yang-Baxter formula via establishing some algebraic concepts. At 2007, W. Rump [2] introduced one of that efforts via employing the concept braces as a expansion the concept of $rad(R)$ the Jacobson radical of a ring R so as for support research involution some results of the Yang Baxter relation. Based on the notion of braces which have been calculated merely for their algebraic characteristics and have been attached to more regard sectors. When the year 2018 came, the author T. Gateva-Ivanova [3] supplied the braces are agreement with braided groups and an involution braiding factor. Moreover, the braces which have also been seen to be much the same as some objects in the concept of group theory. For example, bijective 1-cocycles together with regular subgroups of the holomorph. Indeed, they have been counted in link to the quantum integrable systems, flat manifolds, etc. In addition to that, the contributing via [4] about skew-braces as a generalization of braces, while the author D. Bachiller [5] recognized to uncover that there is a relationship between the two concepts Hopf-Galois theory and the skew braces which was supplementary mature in [6]. These affirmations about the importance of distinguishing braces, which is an orientation authors which have taken in recent times.

Both kinds of the concept of cyclic additive groups with braces were categorized in the references [7,8]. The main idea about braces which have a size pq and p^2q where the two primes p and q were categorized in [9,10]. In actual information, all kinds of braces of the cardinality p^3 which have been substantive via Bachiller [5] while other kinds which are the skew braces whose have the same size done through Nejabati [11]. The idea of the concept of skew-braces were provided in [12]. The authors deal with the left braces as a non-commutative generalization which appeared in [2]. It is turned to generalize radical algebras. As a matter of fact, the connotation of skew braces have been calculated in [6,13]. The elementary motivation of regard in investigation on both notions skew-braces and braces was pushed by the seek for results dismissals for the Yang-Baxter identity. However, there there exist also the nearest link between the thoughts of the skew-braces and Hopf Galois structures, which discussed in Galois extensions of fields. This correlation

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incubated from the invention via [14] of the touch between the concepts radical algebras and regular subgroups which acted of the affine group, with its dependent generalization and employment status to abelian Hopf Galois. This construction on elementary abelian Galois expansion of fields in reference [15].

The main reason of writing this paper is to explorer the action of a new concept which is strongly belong element of the brace $(A, +, \circ)$ and has an ideal I.

2. Background Information

Throughout this paper, A is a set provide with $+$ and $*$ as a two a binary operations is refers to a left brace, when $(A, +)$ represents a group specific abelian group. Also, the distributive combined with associativity be content with the following relations: Suppose the elements $x, y, z \in A$

(i) $(x + y + x * y) * z = x * z + y * z + x * (y * z)$,
(ii) $x * (y + z) = x * y + x * z$.

In addition to that, we define the operation $\circ$ by $x \circ y = x + y + x * y$ where $(A, \circ)$ forms a group. For more information about the original definition view [2]. Also, view the reference [16] to obtain a summarized equivalent definition via employing group theory. Now we will depend on the definition in expressions of the procedure '$\circ$' (see in [16]): A is a set provide with two a binary operations of addition $+$, and multiplication $\circ$ form a brace under the conditions that

(i) $(A, +)$ forms an abelian group,
(ii) $(A, \circ)$ forms a group,
(ii) for every $x, y, z \in A$, $x \circ (y + z) + x = x \circ y + x \circ z$.

We employ the following common determination of an ideal in a brace.

Definition 1 ([17])**.** *Suppose the triple $(A, +, \circ)$ is a brace. Then I acts as an ideal in a brace $(A, +, \circ)$ when $I \subseteq A$. Additionally, for each $x, y \in I$ and each $r \in A$ yield $x + y$ and $x - y$ in I, also $r * x$ to gather $x * r$ in I.*

Named the operation $x \circ y = x * y + x + y$ for all $x, y \in A$. Next step, we will depend on the notation: suppose I and J are subsets of a left brace A. Then $I * J$ form the additive subgroup of A created by elements $i * j$, where $i \in I, j \in J$.

Definition 2 ([6])**.** *Let A be a set provide with two binary operations $+$ and . such that*

(i) $(A, +)$ *forms an abelian group,*
(ii) $(A, .)$ *forms a group,*
(iii) $x(y + z) + x = xy + xz$

for all x, y, and z in A. The above structure form a left brace where $(A, +)$ the additive group and $(A, .)$ the multiplicative group of the left brace.

Under the same axiomatic, we can define a right brace is defined in similar fashion, writing the previous Condition (iii) by the formula $(x + y)z + z = xz + yz$.

It is simple to review and determine that in a left brace A. Here it is necessary to confirm the important relation between the multiplicative group of A which is the multiplicative identity 1 and the neutral element 0 for the additive group of A is equal to each other. To be more convince, the braces A in this paper acts as a left braces.

3. The Main Definition

Definition 3. *Suppose the triple $(A, +, \circ)$ is a brace and I is an ideal of A. An element $a \in I$ is named strongly belong element if $a + b = a$ for all $b \in I$.*

For motivation and a closer look at the previous definition, the following example supplies that.

Example 1. *Assume A form a brace, his elements are complex variables. Suppose* $\delta = x + iy$ *(where x and y are real numbers) is a complex number. Then* $\overline{\delta} = x - iy$ *is conjugate of z Hence,* $\delta + \overline{\delta} = x + iy + x - iy = 2x$*, then the real part is strongly belong element.*

4. The Results

We begin with this result.
Let us recall from ([1], Lemma 2.1) that A is a skew brace. Then $\pi : (A, \circ) \longrightarrow Aut(A, +)$ write as $a \longmapsto \pi_a$, wheresoever $\pi_a(b)$ equal to $-a + a \circ b$. Then, it forms a well-defined group homomorphism.

Proposition 1. *Assume A is a skew brace. Let I and J be ideals of A. If J has a strongly belong membership then* $I + J = J$.

Proof. Suppose a in A, u in I and v in J. Then based on this assumption, we have $\pi_a(u + v) \in I + J$. After that $\pi_a(I + J) \subseteq I + J$. Consequently,
$(u + v) * a = (u \circ \pi_a^{-1} u(v)) * a$ which yields $u * (\pi_a^{-1} u(v) * a) + \pi_a^{-1} u(v) * a + u * a$ belong to $I + J$.
Consequently the previous formula yields
$a \circ (u + v) \circ a' = a + \pi_a((u + v) + (u + v) * a') - a \in I + J$.
Thus it follows that $a \circ (I + J) \circ a' \subseteq I + J$. At the finally $I + J$ forms a normal subgroup of $(A, +)$ since the identity
$a + (\sum_k u_k + v_k) - a$ equal to the identity $(\sum_k u_k + v_k)$, after remove the term a.
Due to u_k in I and v_k in J for all k with employing the fact that J strongly belong membership, we deduce that
$a + (\sum_k u_k + v_k) - a$ equal to the identity $(\sum_k u_k + v_k) = v_k = J$. □

Theorem 1. *The sum of any number of N-ideals in A equal to an ideal has strongly belong membership in A, where A acts as a brace.*

Proof. Suppose I and J are two N-ideals belong to A. First of all, we move to provide the relation $I + J$ equal to the set $\{a + b : a \in I, b \in J\}$ forms an n-ideal in I. Suppose $a_1, a_2, ...$ is an N-sequence with the relation $a_1 = a + b$, where a in I and b in J.
Take into consideration the two results. The first one, factor brace A/J while the second is the N-sequence,
$a_1' = a_1 + J = a + J, a_2' = a_2 + J, ...$ in A/J. We observe that every N-sequence in A starting with $a \in I$ will approach to zero. Consequently, every n-sequence in A/J will approach to zero based on $(r + J) * (s + J)$ equal to $r * s + J$ for all $r, s \in A$.
In ([18], Question 2.1(2)) for any brace such that the operation defined as $a * b = -a + a \circ b - b$ is associative is a two-sided brace. This fact is true.
Employing this relation in $(r + J) * (s + J) = r * s + J$, where
$r * s + J = -r + r \circ s - s + J$ equal to $-r + (r \circ s) - s + J$. Writing this relation to
$-r + (r \circ s) - s + J = -r + (r \circ (s + 0)) - s + J$.
New applying the relation $a \circ (b + c) = a \circ b - a + a \circ c$ to the term $(r \circ (s + 0))$, we arrive to
$-r + (r \circ (s + 0)) - s + J$ equal to $-r + (r \circ s) - r + (r \circ 0) - s + J$.
Moreover,
$-r + (r \circ s) - s + J$ yields $-r + (r \circ s) - r + (r \circ 0) - s + J$.
Obviously, we can remove some terms from this relation
$r = (r \circ 0)$. Based on true certainty relation $a \circ b = a * b + a + b$, we harvest that
$r = r * 0 + r + 0$. Thus, we conclude that
$A * 0 = 0$. Hence, if $s = 0$, then the relation $(r + J) * (0 + J)$ which equal to the relation $r * 0 + J = J$. □

The following result is immediate consequences of the previous theorem.

Corollary 1. *Assume A is a brace, such that A has sum of any number of N-ideals then $A * 0 = 0$.*

Proposition 2. *Assume A is a skew brace, J is an ideal has strongly belong membership in A. Suppose I is an ideal of skew brace A/J such that $I^\star$ equal to $\{a \in A : a + J \in I\}$ is an ideal in A. Then $\pi_J(J)$ in I.*

Proof. Observe that $a \in I^\star$ if and only if $a + J \in I$. Let $a, b \in I^\star$. Due to the fact $a + J$ in I and $b + J$ in I. Then $a + b + J$ equal to $(a + J) + (b + J)$ which belong to I. Consequently, $a + b \in I^\star$. In like manner, if $a \in I^\star$ and $c \in A$, then $a + J$ in I. Therefore, $\pi_c(a) + J = \pi_{c+J}(a + J)$ in I. Based on J has strongly belong membership in A, then $\pi_J(J) \in I$. This step completes the proof. □

Corollary 2. *Assume $J \subseteq I$ are ideals in a skew brace A such that J has strongly belong membership in A. Then I/J forms an ideal in A/J and $J \subseteq I/J$.*

Proof. Suppose $a + J$ and $b + J$ in I/J and c in A. Then $a \in I$ and $b \in I$. Consequently, by reason J is an ideal, $(a + J) + (b + J)$ equal to $a + b + J$ which belong to I/J. Then $\pi_{c+J}(a + J)$ equal to $\pi_{c(a)} + J$ which lie in I/J. Due to the hypothesis J has strongly belong membership in A, then the last expression modifies to $J \subseteq I/J$.This completes the proof. □

Depend on the fact that $I + J$ forms an ideal, we find the following corollary.

Corollary 3. *Assume A is a skew brace, I and J are ideals in A such that J has strongly belong membership in A, then $(I + J)/J$ is an ideal in A/J.*

Note: In [19], suppose A acts as a left brace. For any arbitrary element $a \in A$, we define a function $\pi_a : A \longrightarrow A$ via $\pi_a(b) = ab - a$, where $b \in A$. Based on this function, we find the following result.

Proposition 3. *A is a left brace. Then*

(i) $\pi_a(x + y) = a(x + y - 2)$, *holds for all $a, x, y \in A$,*
(ii) $\pi_a \pi_b = a(b(x - 1) - 1)$, *holds for all $a, b, x \in A$.*

Proof. (i) Let a, x, and y in A. Then $\pi_a(x + y) = a(x + y) - a = ax + ay - a - a$ which yields $a(x + y) - 2a = a(x + y - 2)$.
(ii) Let a, b and x in A. Then $\pi_a \pi_b(x) = \pi_a(bx - b)$ which yields $a(bx - b) - a = a(b(x - 1) - 1)$.
By this step The proof is complete. □

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