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The Advanced Boundary Integral Equation Method for Modelling Wave Propagation in Layered Acoustic Metamaterials with Arrays of Crack-Like Inhomogeneities †

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Abstract: The three-dimensional problem of the modelling of elastic wave propagation in a multi-layered acoustic metamaterial, which is a periodic elastic composite with periodic arrays of interface cracks or planar voids of an arbitrary shape, is considered. The boundary integral equation method is extended for this purpose. The unknown crack opening displacement vectors for each array are related using the Floquet theorem and solved using the Galerkin method at the reference delaminations in the arrays. The developed method provides an efficient tool for fast parametric analysis of the influence of the characteristics of the periodic arrays of cracks on the transmission and diffraction of elastic waves. Two modifications of the boundary integral equation method are proposed and compared for rectangular cracks. To reduce computational costs, a preliminary analytical evaluation of the arising integral representations in terms of the Fourier transform of Green's matrices and the crack opening displacements is presented.

Keywords: wave propagation; boundary integral equation method; spectral method; acoustic metamaterial; crack; void; periodic composite; diffraction.

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1. Introduction

A novel class of composites, the so-called acoustic metamaterials (AMMs), which provide advanced characteristics attracting special attention of the researchers last decades [1,2]. AMMs reproduce unique properties that open up prospects for passive and active wave energy manipulation. At the moment, various AMMs have already been developed with a scope in ultrasonic technology, acoustoelectronics, hydroacoustics, architectural acoustics, sound absorption [3,4].

AMMs typically have a periodic or quasi-periodic structure, where arrays of inhomogeneities such as holes, voids, or inclusions are embedded in a matrix that can also be a composite. The mathematical modeling is usually performed at the first stages of the design of new AMMs to select the structure parameters that provide desired wave properties. In this study, multi-layered AMM with doubly periodic arrays of delaminations/cuts at some interfaces is considered. To describe the dynamic behavior of the considered class of AMMs, a modification of the boundary integral equations method (BIEM) proposed by Glushkov and Glushkova [5] is developed. A similar employment of the BIEM was proposed in [6], where the propagation of plane waves through the interface of two elastic media with double periodic array of interface cracks was considered. In this study, the advanced BIEM is presented to simulate wave motion in a multi-layered AMM with multiple doubly periodic arrays of cracks or voids.

2. Statement of the Problem

The problem of elastic waves propagation in multi-layered AMM composed of N periodically arranged unit-cells made of two elastic isotropic layers is considered. M doubly periodic arrays of cracks or infinitesimally thin voids are situated at the interfaces and it's formed a rectangular lattice. It is assumed that the periodic stack of layers is located between two elastic half-spaces and a plane wave comes from the lower half-space at a certain angle to the interfaces. For convenience, the Cartesian coordinates $\{x_1, x_2, x_3\}$ are introduced so that interfaces are parallel to x_1Ox_2 the plane and the voids are situated along axes Ox_1 and Ox_2. An example of the AMM with $M = 2$ doubly periodic arrays is shown in Figure 1. Accordingly, $V_0 = \{x_3 \le 0\}$ is the lower half-space, and $V_{2N+1} = \{x_3 > h_{2N}\}$ is the upper half-space. The unit-cell consists of two components and, therefore, totally $2N$ layers $V_k = \{|x_1| < \infty, |x_2| < \infty, h_{k-1} < x_3 \le h_k\}$ are considered. Each infinite three-dimensional layer V_k is made of homogeneous, isotropic material with the mass density ρ_k, Young's modulus E_k and Poisson's ratio ν_k.

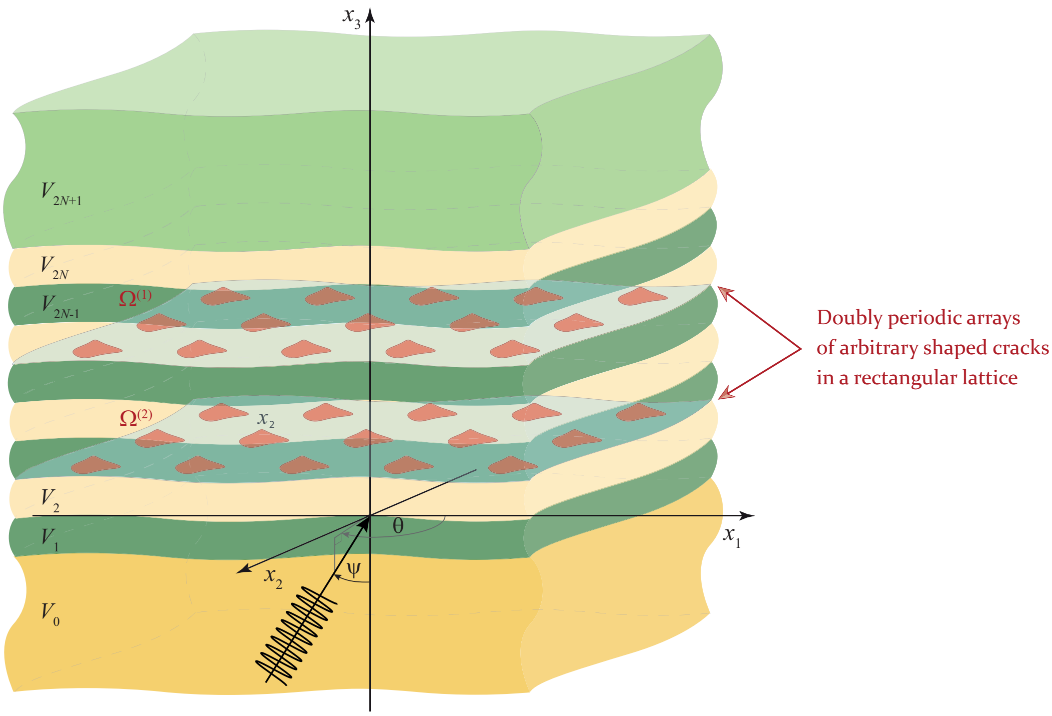

Figure 1. The geometry of the problem: multi-layered elastic periodic composite with two doubly periodic arrays of cracks (with rectangular lattice).

Multiple doubly periodic arrays $\Omega^{(m)}$ of cracks or voids with the same spacing between the centers of cracks are situated in the planes $x_3 = d^{(m)}$, $m = \{1, 2\}$. The rectangular lattice corresponding to each doubly periodic array $\Omega^{(m)}$ is based on the vectors $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ and has dimensions s_1, s_2 of the unit-cell as shown in Figure 2a. In accordance with the location of the cracks, the whole media can be considered as a doubly periodic array of unbounded parallelepipeds $\bigcup_{j_1,j_2} G^{(m)}_{j_1 j_2} = \{|x_1| \le s_1, |x_2| \le s_2, |x_3| < \infty\}$, which allows to describe scattering by all doubly periodic arrays. The intersection of the parallelepiped

$$G^{(m)}_{00} = \{\boldsymbol{x}|\,(x_1, x_2) = \beta_1 \boldsymbol{g}_1 + \beta_2 \boldsymbol{g}_2, \quad |x_3| < \infty\}, \quad \beta_i \in [-1/2, 1/2]$$

with the plane $x_3 = d^{(m)}$ is chosen as a reference unit-cell in m-th array containing the reference crack $\Omega^{(m)}_{00}$. The centre of the reference crack $\Omega^{(m)}_{00}$ for each array is assumed at the origin of Cartesian coordinates. Geometrical sizes of the unit-cell are denoted as

$$s_1 = |\boldsymbol{g}_1|, \qquad s_2 = |\boldsymbol{g}_2|,$$

whereas the centre of the unit-cell $G_{ij}^{(m)}$ is defined by the vector

$$\boldsymbol{a}_{j_1j_2}^{(m)} = \{x_{j_1j_2}^{(m)}, y_{j_1j_2}^{(m)}, d^{(m)}\} = \{s_1j_1, s_2j_2, d^{(m)}\}.$$

The center of crack-like voids $\Omega_{j_1j_2}^{(m)}$ is shifted from the centre of the unit-cell by vector $\boldsymbol{b}^{(m)}$ as shown in Figure 2b.

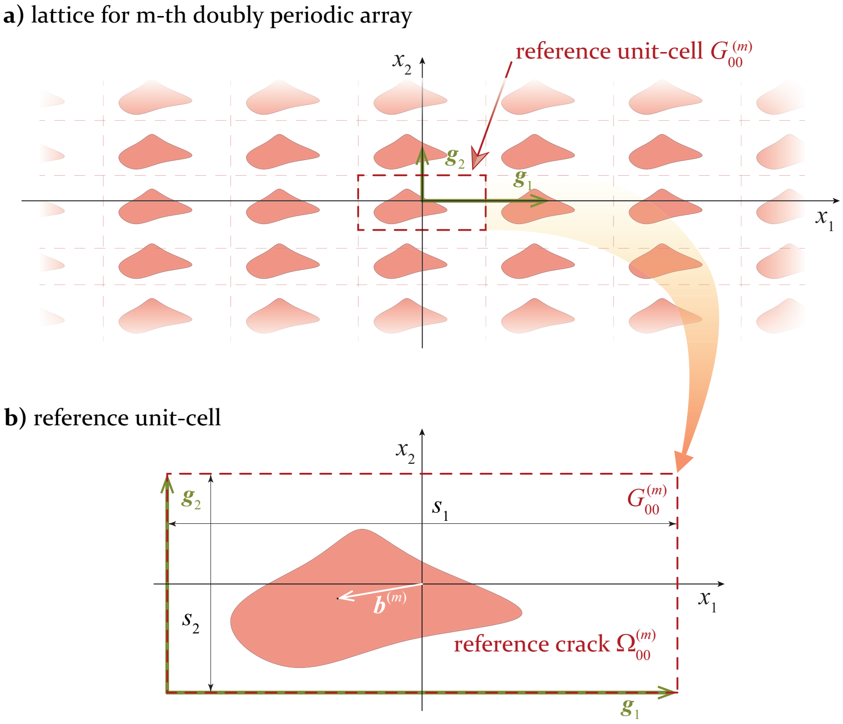

Figure 2. m-th doubly periodic array of rectangular cracks: (**a**) Lattice for m-th doubly periodic array. (**b**) the reference unit-cell.

The steady-state harmonic motion of the multi-layered periodic elastic structure with the circular frequency ω is governed by the Lame-Navier equation with respect to the displacement vector $\boldsymbol{u}$. The displacement vector $\boldsymbol{u}$ and the traction vector $\boldsymbol{\tau} = \mathbf{T}_3[\boldsymbol{u}] = (\sigma_{13}, \sigma_{23}, \sigma_{33})$ are assumed continuous outside the voids $\Omega_{i,j}^{(m)}$, while stresses and displacements are related by the Hooke's law. The stress-free boundary conditions are assumed at the crack faces so that unknown crack opening displacement (COD) function $\Delta\boldsymbol{u}^{(m)}(\boldsymbol{x})$ is introduced for each plane $x_3 = d^{(m)}$ containing m-th doubly periodic array.

3. The Advanced Boundary Integral Equation Method

Let us consider plane wave scattering propagating in the composite by M arrays. In this case, the wave-field $\boldsymbol{u}^0$ incident by a plane wave incoming from the lower half-space V_0 can be simulated using the transfer matrix method [7]. The total wave-field in the composite is the sum of the incident wave-field $\boldsymbol{u}^0$ propagating in the layered structure in the absence of inhomogeneities and the wave-fields $\tilde{\boldsymbol{u}}_{j_1,j_2}^{m}$ scattered by each crack in the doubly periodic arrays $\boldsymbol{\Omega}_m = \bigcup_{j_1,j_2} \Omega_{j_1j_2}^{(m)}$.

The mutual effect of the cracks on each other can be taken into account using the Floquet theorem. Therefore, the two-dimensional Fourier transform of the COD $\Delta\boldsymbol{U}$ with parameter $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$ has the following representation

$$\Delta\boldsymbol{U}^{(m)}(\boldsymbol{\alpha}) = \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}_{j_1 j_2}^{(m)}(\boldsymbol{\alpha}) = \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}_{00}^{(m)}(\boldsymbol{\alpha})\, \mathrm{e}^{\mathrm{i}\boldsymbol{a}_{j_1 j_2}^{(m)} \cdot \boldsymbol{\alpha}_p}, \tag{1}$$

where $\boldsymbol{\alpha}_p = k_0 \boldsymbol{p} + \boldsymbol{\alpha}$, $\boldsymbol{p} = \{p_1, p_2\}$ is the two-dimensional projection of the unit vector of the wave propagation vector on the plane $x_3 = d^{(m)}$ and k_0 is the wave-number of the incident plane wave with polar and azimuthal incidence angles θ and ϕ.

On the other hand, the scattered field can be expressed in terms of the two-dimensional Fourier transform in accordance with the BIEM [5,6] as contour integrals along contours Γ_i bending poles and branch points of the two-dimensional Fourier transform of Green's matrix of the whole structure constructed in the same manner as [8]. Notice that the Fourier transform of the unknown traction vector can be expressed in terms of the COD.

The substitution of the integral representation for total wave-field into the stress-free boundary conditions and taking into account the Hooke's law and the Floquet theorem gives the following boundary integral equation for the reference cracks in the m-th array:

$$\frac{1}{4\pi^2} \sum_{i=1}^{M} \int_{\Gamma_1} \int_{\Gamma_2} \tilde{\mathbf{S}}^{(i)}(\boldsymbol{\alpha}^{j_1 j_2}, d^{(m)}) \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}_{00}^{(i)}(\boldsymbol{\alpha}^{j_1 j_2})\, \mathrm{e}^{\mathrm{i}\boldsymbol{a}_{j_1 j_2}^{(m)} \cdot \boldsymbol{\alpha}_p} \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{y}} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 = -\boldsymbol{\tau}^0(\boldsymbol{x}), \boldsymbol{x} \in \Omega_{00}^{(m)}$$

$$\boldsymbol{\alpha}^{j_1 j_2} = \left\{-k_0 p_1 + \frac{2\pi j_1}{s_1}, -k_0 p_2 + \frac{2\pi j_2}{s_2}\right\}, \qquad \boldsymbol{y} = \{x_1, x_2\}. \tag{2}$$

For more details related to the derivation of the boundary integral equation and about $\tilde{\mathbf{S}}^{(i)}$ see [9].

Boundary integral equation (2) is solved using the Galerkin scheme. The unknown COD for the crack $\Omega_{00}^{(m)}$ in the reference unit-cell $G_{00}^{(m)}$ is approximated by the complete set of basis functions $\varphi_k(x_1, x_2)$:

$$\Delta\boldsymbol{u}_{00}^{(m)}(x_1, x_2) = \sum_{k=1}^{\infty} \boldsymbol{c}_k^{(m)} \varphi_k^{(m)}(x_1, x_2). \tag{3}$$

The choice of basis and projection functions depends on the cracks shape. In the case of rectangular cracks, the CODs can be expanded in terms of the Chebyshev polynomials $\mathrm{U}_n(x)$ of the second kind with the square root weight $p_k(x) = \mathrm{U}_{k-1}(x)\sqrt{1-x^2}$ for each coordinate. For arbitrary shaped cracks/voids the unknown COD vector is expanded in terms of axisymmetric basis functions

$$\phi(x_1, x_2) = \begin{cases} \left(1 - x_1^2 - x_2^2\right)^{\pi-1}, & x_1^2 + x_2^2 < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Though convergence of the COD is not guaranteed in a continuous metric, the COD convergence of the solution at the nodal points for all $h > 0$ is guaranteed [5] .

As a result of the application of the Bubnov–Galerkin scheme to (2) keeping N terms after reduction, the following system is obtained:

$$\sum_{m=1}^{M} \sum_{k=1}^{N} \mathbf{A}_{jk}^{(m)} \cdot \boldsymbol{c}_k = \boldsymbol{f}_j, \qquad j = 1, 2 \cdots M. \tag{4}$$

The right-hand side of system (4) is the projection of the wave-field $\boldsymbol{\tau}^0$ onto the projection functions $\psi_j(x_1, x_2)$, whereas double series

$$\mathbf{A}_{jk}^{(m)} = \frac{1}{s_1 s_2} \sum_{j_1=-M_1}^{M_1} \sum_{j_2=-M_2}^{M_2} \tilde{\mathbf{S}}^{(i)}(\boldsymbol{\alpha}^{j_1 j_2}, d^{(m)}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}^{j_1 j_2} \cdot \boldsymbol{b}^{(m)}} \Phi_j^*((\boldsymbol{\alpha}^{j_1 j_2})^*) \Psi_j^*((\boldsymbol{\alpha}^{j_1 j_2})^*) \tag{5}$$

describes the scattering by m-th array the wave-field induced due to the presence of j-th array. Here Φ_j and Ψ_j are the two-dimensional Fourier transforms of the basis and projection functions respectively.

The calculation of the left-hand side of system (4) demands computations of double series (5), which exhibit a low convergence rate for rectangular cracks if Chebyshev polynomials are employed as basis and projection functions due to the Fourier transform $\Phi_k(\alpha_j) \sim \alpha_j^{-3/2}$ as well the kernel $\tilde{\mathbf{S}}^{(i)}(\alpha_1, \alpha_2) \sim \alpha$ at $\alpha \to \infty$, thus the double series summarize products of four Bessel functions and power function. The convergence of series (5) is shown estimating the absolute values. Moreover, such analytic evaluation allows for to determine direction in the $\boldsymbol{\alpha}$-plane, where the slowest convergence is observed (along axes $O\alpha_1$ and $O\alpha_2$). It is shown that the terms with $\{\alpha_1, \alpha_2\}$ lying inside a certain astroid and along the coordinate axes provide the largest contribution to the sum, which is used for calculating double series.

Figure 3 illustrate the convergence of several non-zero components of the matrices $\mathbf{A}_{jk}^1$ at lower and higher frequencies $k_0 s_2 = 2$ (Figure 3a) and $k_0 s_2 = 10$ (Figure 3b), where k_0 is the wavenumber of incoming plane longitudinal wave. The variation of the relative error during the double series calculation

$$\epsilon_{\mathrm{r}}\left(A_{jk;ij}\right) = \frac{A_{jk;ij} - A_{jk;ij}^{(\mathrm{exact})}}{|A_{jk;ij}^{(\mathrm{exact})}|}$$

with respect to the number of terms M_i is presented here. $A_{jk_1k_2;ij}^{(\mathrm{exact})}$ is calculated numerically setting $M_1 = M_2 = 2 \times 10^4$. The higher frequency, the greater convergence rate of the double series. The latter can be explained by the fact that the Fourier transform of the kernel of the boundary integral equation $\mathbf{S}(\alpha_1, \alpha_2)$ decreases slowly at lower frequencies ω. For non-square rectangular cracks ($l_1 \neq l_2$), the ratio between the numbers of terms N_2/N_1 should be approximately equal to the ratio l_2/l_1.

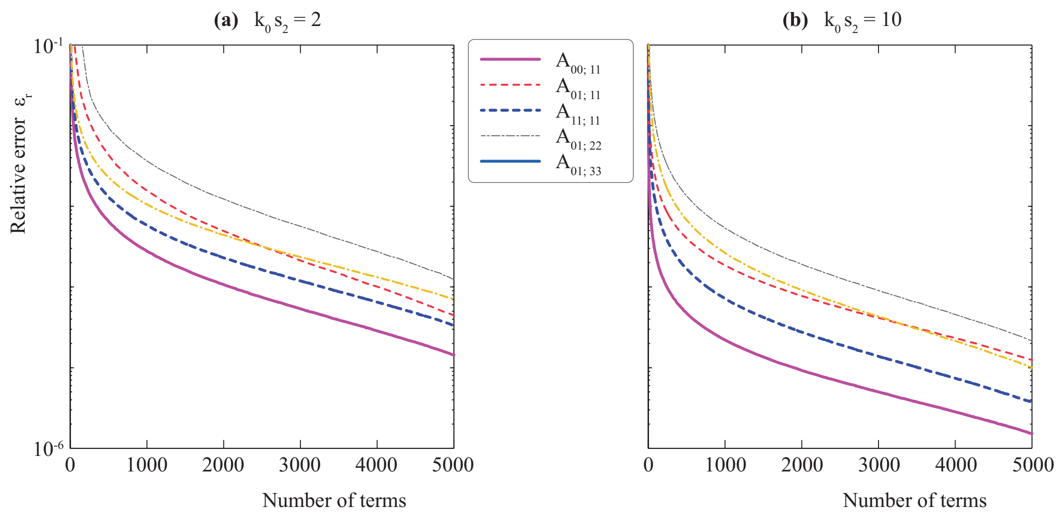

Figure 3. The convergence of the double series

4. Conclusions

Two choices of basis and projection functions have been compared. In the case of the same basis and projection functions in the Bubnov-Galerkin method, a slow convergence of the arising double series is observed. If axisymmetric functions proposed by Glushkov and Glushkova [10] are used as projection functions in Petrov-Galerkin method, which guarantees fast convergence in series, then more basis functions are demanded for better accuracy in the crack opening displacements. The results of the numerical analysis show a good accuracy and convergence rate of the proposed method. The authors believe that the proposed advanced BIEM will be further employed for experimental and theoretical studies of the wave propagation in acoustic metamaterials with doubly periodic arrays of crack-like voids, see e.g., [11].

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