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Banach Fixed Point Theorem in Extended $b_v(s)$- Metric spaces

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Abstract: We define the class of extended $b_v(s)$-metric spaces by replacing the real number $s \geq 1$ with strictly increasing continuous function ϕ in the definition of $b_v(s)$-metric spaces introduced by Mitrović and Radenović (2017). Also, we presented an example for this newly introduced space and exhibited that, in a particular situation, the class of extended $b_v(s)$-metric spaces reduces to the class of $b_v(s)$-metric spaces. Then after, we establish a fixed point theorem which ensured the existence of a fixed point for the self map satisfying the Banach contractive condition in the context of this newly defined space. Moreover, we compared the proved result with the existing fixed point theorems in literature.

Keywords: b-metric space; rectangular b- metric space; $b_v(s)$-metric space; Banach contraction principle

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1. Introduction and Preliminaries

As the solution of certain problems of nonlinear analysis relies heavily on fixed-point theory, one of the key findings of metric fixed-point theory, the Banach contraction principle, is the subject of intense investigation. For the last three decades, the generalization of the metric space is one of the key directions in which the contraction principle is continuously flourishing. In 1989, Bakhtin [1] pioneered the idea of b-metric space while moving in the same direction that was conventionally defined in 1993 by Czerwik [2]. On the other side, Branciari [3] generalized the metric spaces by replacing the triangle inequality with the quadrilateral inequality, and later on it was referred to as rectangular metric space by the researchers. In 2015, George et al.[4] defined the class of rectangular b-metric spaces as an extension of the class of b-metric spaces. By the ideas of Bakhtin [1] and George et al.[4], a notion of $b_v(s)$- metric space is announced by Mitrović and Radenović [5] in the recent past, which in general is an extension of v-generalized metric spaces, b-metric spaces and rectangular b-metric spaces.

Very recently, Mustafa et al.[6] presented a new notion, namely extended rectangular b-metric space, as an extension of rectangular b-metric spaces, inspired by the idea of Parvaneh [7] for defining the concept of extended b-metric spaces. In a similar manner, we here introduce the concept of extended $b_v(s)$-metric spaces and ensure the existence of a fixed point for the self-map that satisfy the Banach contractive condition, i.e., we present the analogue of the Banach contraction principle in this newly defined space.

Here, we give a few definitions and findings that will be important in the discussion that follows.

Definition 1. *[4] Let Ω be a nonempty set and $s \geq 1$ be a real number. A mapping $d_{rb} : \Omega \times \Omega \to [0, \infty)$ is said to be a rectangular b-metric on Ω, if the following axioms hold for all $x, y \in \Omega$.*

1. $d_{rb}(x, y) = 0$ *if and only if* $x = y$,
2. $d_{rb}(x, y) = d_{rb}(y, x)$,

3. *$d_{rb}(x,y) \leq s[d_{rb}(x,\omega_1) + d_{rb}(\omega_1,\omega_2) + d_{rb}(\omega_2,y)]$ for all distinct points $\omega_1,\omega_2 \in \Omega - \{x,y\}$.*

The pair (Ω, d_{rb}) is called a rectangular b-metric space.

Mustafa et al.[6] generalized rectangular b-metric space by utilizing the following class of strictly increasing continuous functions defined by Parveneh [7].

$$\Psi = \{\phi : [0,\infty) \to [0,\infty), \phi \text{ is strictly increasing continuous function} \\ \text{and } t \leq \phi(t) \text{ with } \phi(0) = 0\}$$

Definition 2. *[6] Let Ω be a nonempty set and $\phi \in \Psi$. A mapping $d^e_{rb} : \Omega \times \Omega \to [0,\infty)$ is said to be an extended rectangular b-metric on Ω, if the following axioms hold for all $x,y \in \Omega$.*

1. *$d^e_{rb}(x,y) = 0$ if and only if $x = y$,*
2. *$d^e_{rb}(x,y) = d^e_{rb}(y,x)$,*
3. *$d^e_{rb}(x,y) \leq \phi[d^e_{rb}(x,\omega_1) + d^e_{rb}(\omega_1,\omega_2) + d^e_{rb}(\omega_2,y)]$ for all distinct points $\omega_1,\omega_2 \in \Omega - \{x,y\}$.*

The pair (Ω, d^e_{rb}) is called an extended rectangular b-metric space.

The concept of $b_v(s)$-metric spaces given by Mitrović and Radenović [5] (also called v-generalized b-metric spaces in Kadyan et al. [8]) is defined as under:

Definition 3. *[5] Let Ω be a nonempty set and $s \geq 1$ be a real number. A mapping $d_{v_g b} : \Omega \times \Omega \to [0,\infty)$ is said to be a $b_v(s)$-metric on Ω, if the following axioms hold for all $x,y \in \Omega$.*

1. *$d_{v_g b}(x,y) = 0$ if and only if $x = y$,*
2. *$d_{v_g b}(x,y) = d_{v_g b}(y,x)$,*
3. *$d_{v_g b}(x,y) \leq s[d_{v_g b}(x,\omega_1) + d_{v_g b}(\omega_1,\omega_2) + ... + d_{v_g b}(\omega_v,y)]$ for all distinct points $\omega_1,\omega_2,...,\omega_v \in \Omega - \{x,y\}$, where v is a natural number.*

The pair $(\Omega, d_{v_g b})$ is called a $b_v(s)$-metric space.

2. Main Result

We start by introducing the concept of extended $b_v(s)$-metric spaces.

Definition 4. *Let Ω be a nonempty set and $\phi \in \Psi$. A mapping $d_{v_g b} : \Omega \times \Omega \to [0,\infty)$ is said to be an extended $b_v(s)$-metric on Ω, if the following axioms hold for all $x,y \in \Omega$.*

1. *$d^e_{v_g b}(x,y) = 0$ if and only if $x = y$,*
2. *$d^e_{v_g b}(x,y) = d^e_{v_g b}(y,x)$,*
3. *$d^e_{v_g b}(x,y) \leq \phi[d^e_{v_g b}(x,\omega_1) + d^e_{v_g b}(\omega_1,\omega_2) + ... + d^e_{v_g b}(\omega_v,y)]$ for all distinct points $\omega_1,\omega_2,...,\omega_v \in \Omega - \{x,y\}$, where v is a natural number.*

The pair $(\Omega, d^e_{v_g b})$ is said to be an extended $b_v(s)$-metric space.

Remark 5. *In respect of this newly introduced space, it is important to note that*

(1) *if $v = 1$, then it reduces to extended b-metric spaces.*
(2) *if $v = 2$, then it reduces to extended rectangular b-metric spaces.*
(3) *if we define $\phi(t) = st$, where $s \geq 1$, then it reduces to $b_v(s)$-metric spaces.*
(3) *if we take $\phi(t) = t$, then it reduces to v-generalized metric spaces.*

In the light of above remark, it is clear to mention that the class of extended $b_v(s)$-metric spaces is larger than the class of extended rectangular b-metric space and the class of $b_v(s)$-metric spaces.
Now, we construct an example for the extended $b_v(s)$-metric space.

Example 6. *Let (Ω, d_{vg}) is a v-generalized metric space. Define $\rho(x,y) = e^{d_{vg}(x,y)} - e^{-d_{vg}(x,y)}$. We will show that (Ω, ρ) is an extended $b_v(s)$-metric space with $\phi(x) = e^x - e^{-x} \ \forall\, x \geq 0$. It is easy to verify conditions 1 and 2 of Definition 4. Now, we verify the condition 3 as follows:*

$$\begin{aligned}
\rho(x,y) &= e^{d_{vg}(x,y)} - e^{-d_{vg}(x,y)} \\
&\leq e^{d_{vg}(x,\omega_1)+d_{vg}(\omega_1,\omega_2)+\cdots+d_{vg}(\omega_v,y)} - e^{-(d_{vg}(x,\omega_1)+d_{vg}(\omega_1,\omega_2)+\cdots+d_{vg}(\omega_v,y))} \\
&\leq e^{\left[e^{d_{vg}(x,\omega_1)}-e^{-d_{vg}(x,\omega_1)}+e^{d_{vg}(\omega_1,\omega_2)}-e^{-d_{vg}(\omega_1,\omega_2)}+\cdots+e^{d_{vg}(\omega_v,y)}-e^{-d_{vg}(\omega_v,y)}\right]} \\
&\quad -e^{-\left[e^{d_{vg}(x,\omega_1)}-e^{-d_{vg}(x,\omega_1)}+e^{d_{vg}(\omega_1,\omega_2)}-e^{-d_{vg}(\omega_1,\omega_2)}+\cdots+e^{d_{vg}(\omega_v,y)}-e^{-d_{vg}(\omega_v,y)}\right]} \\
&\leq e^{[\rho(x,\omega_1)+\rho(\omega_1,\omega_2)+\cdots+\rho(\omega_v,y)]} - e^{-[\rho(x,\omega_1)+\rho(\omega_1,\omega_2)+\cdots+\rho(\omega_v,y)]} \\
&= \phi[\rho(x,\omega_1)+\rho(\omega_1,\omega_2)+\cdots+\rho(\omega_v,y)].
\end{aligned}$$

Thus, all the conditions of Definition 4 are satisfied and hence (Ω, ρ) is and extended $b_v(s)$-metric space.

The convergence of a sequence and Cauchy sequence in an extended $b_v(s)$-metric space is defined in usual sense as mentioned under.

Definition 7. *Let $(\Omega, d^e_{v_g b})$ be an extended $b_v(s)$-metric space. A sequence $\zeta_n \in \Omega$ is said to be*

(1) a convergent sequence (converges to a point $\zeta \in \Omega$) if for each $\epsilon > 0, \exists$ a positive integer N such that $d^e_{v_g b}(\zeta_n, \zeta) < \epsilon$ for all $n \geq N$. Symbolically, it may be written as $\zeta_n \to \zeta$ whenever $n \to \infty$.

(2) a Cauchy sequence, if for each $\epsilon > 0, \exists$ a positive integer N such that $d^e_{v_g b}(\zeta_n, \zeta_{n+p}) < \epsilon$ for all $n \geq N$ and $p > 0$. It is also denoted as $\lim_{n\to\infty} d^e_{v_g b}(\zeta_n, \zeta_{n+p}) = 0$.

The extended $b_v(s)$-metric space $(\Omega, d^e_{v_g b})$ is said to be complete if every Cauchy sequence in Ω converges in Ω itself. Now, we establish the following lemmas which are required to prove our forthcoming result.

Lemma 8. *Let $(\Omega, d^e_{v_g b})$ be an extended $b_v(s)$-metric space and $\{\zeta_n\}$ be a sequence of distinct points in Ω. Further, if $d^e_{v_g b}(\zeta_m, \zeta_n) \leq \eta d^e_{v_g b}(\zeta_{m-1}, \zeta_{n-1})\ \forall m, n$, where $\eta \in [0, 1)$. Then the sequence $\{\zeta_n\}$ is Cauchy.*

Proof. As $d^e_{v_g b}(\zeta_m, \zeta_n) \leq \eta d^e_{v_g b}(\zeta_{m-1}, \zeta_{n-1})$ for all $m, n \in \mathbb{N}$, it follows that

$$\begin{aligned}
d^e_{v_g b}(\zeta_{m+k}, \zeta_{n+k}) &\leq \eta d^e_{v_g b}(\zeta_{m+k-1}, \zeta_{n+k-1}) \\
&\leq \eta^2 d^e_{v_g b}(\zeta_{m+k-2}, \zeta_{n+k-2}) \\
&\ \vdots \\
&\leq \eta^k d^e_{v_g b}(\zeta_m, \zeta_n) \text{ for all } m, n. \qquad (2.1)
\end{aligned}$$

Further, it is obvious to say that

$$d^e_{v_g b}(\zeta_n, \zeta_{n+1}) \leq \eta^n d^e_{v_g b}(\zeta_0, \zeta_1) \text{ for all } n. \qquad (2.2)$$

Using condition (3) of Definition 4, we have

$$\begin{aligned} d^e_{v_g b}(\zeta_n,\zeta_{n+p}) &\leq \phi[d^e_{v_g b}(\zeta_n,\zeta_{n+1}) + d^e_{v_g b}(\zeta_{n+1},\zeta_{n+2}) + \cdots \\ &\quad + d^e_{v_g b}(\zeta_{n+v-1},\zeta_{n+v}) + d^e_{v_g b}(\zeta_{n+v},\zeta_{n+p})] \\ &\leq \phi[\eta^n d^e_{v_g b}(\zeta_0,\zeta_1) + \eta^{n+1} d^e_{v_g b}(\zeta_0,\zeta_1) + \cdots \\ &\quad + \eta^{n+v-1} d^e_{v_g b}(\zeta_0,\zeta_1) + \eta^n d^e_{v_g b}(\zeta_v,\zeta_p)] \\ &= \phi\left[\frac{\eta^n(1-\eta^v)}{1-\eta} d^e_{v_g b}(\zeta_0,\zeta_1) + \eta^n d^e_{v_g b}(\zeta_v,\zeta_p)\right] \end{aligned} \tag{2.3}$$

Since $\eta \in [0,1)$, then we concluded that $\lim_{n\to\infty} d^e_{v_g b}(\zeta_n,\zeta_{n+p}) \leq \phi(0) = 0$. Thus the sequence $\{\zeta_n\}$ is Cauchy. □

In an extended $b_v(s)$-metric space, the sequence may converges to more than one point (see Example 2 of [8]). The following lemma ensures that Cauchy sequence in an extended $b_v(s)$-metric space converges to at most one point in the given space.

Lemma 9. *Let $(\Omega, d^e_{v_g b})$ be an extended $b_v(s)$-metric space and $\{\zeta_n\}$ be a Cauchy sequence of distinct points in Ω.Then $\{\zeta_n\}$ can converge to at most one point.*

Proof. Suppose that the sequence ζ_n converges to two distinct point of Ω say ζ and $\zeta*$. Then, there exists a real number $r > 0$ such that for all $n \geq r$, the terms of the sequence $\{\zeta_n\}$ are distinct from ζ and $\zeta*$. Let $\epsilon > 0$ be given, then there exist positive integers l_1, l_2, l_3 (as the sequence $\{\zeta_n\}$ is Cauchy and converges to ζ and $\zeta*$) such that

$$d^e_{v_g b}(\zeta_n,\zeta_m) < \frac{\epsilon}{(v+1)} \ \ \forall\, n,m \geq l_1,$$

$$d^e_{v_g b}(\zeta_n,\zeta) < \frac{\epsilon}{(v+1)} \ \ \forall\, n \geq l_2,$$

and

$$d^e_{v_g b}(\zeta_n,\eta) < \frac{\epsilon}{(v+1)} \ \ \forall\, n \geq l_3.$$

Define $l = \max\{r, l_1, l_2, l_3\}$. Then,

$$\begin{aligned} d^e_{v_g b}(\zeta,\zeta^*) &\leq \phi[d^e_{v_g b}(\zeta,\zeta_{n+v-2}) + d^e_{v_g b}(\zeta_{n+v-2},\zeta_{n+v-3}) + \cdots \\ &\quad + d^e_{v_g b}(\zeta_{n+1},\zeta_n) + d^e_{v_g b}(\zeta_n,\zeta_l) + d^e_{v_g b}(\zeta_l,\zeta^*)] \\ &< \phi\left[\frac{\epsilon}{(v+1)} + \frac{\epsilon}{(v+1)} + \cdots + \frac{\epsilon}{(v+1)}\right] = \phi(\epsilon), \ \text{ for } \ n \geq l. \end{aligned}$$

That implies $\phi^{-1}(d^e_{v_g b}(\zeta,\zeta*)) < \epsilon$ and hence $\zeta = \zeta^*$ as $\epsilon > 0$ is arbitrary and $\phi(0) = 0$. So, the sequence $\{\zeta_n\}$ converges to a unique point in Ω. □

Now, we prove the Banach fixed point theorem in extended $b_v(s)$-metric space.

Theorem 10. *Let $(\Omega, d^e_{v_g b})$ be an extended $b_v(s)$-metric space and $T : \Omega \to \Omega$ be a mapping satisfying*

$$d^e_{v_g b}(T\alpha, T\beta) \leq \eta d^e_{v_g b}(\alpha,\beta) \text{ for all } \alpha,\beta \in \Omega, \tag{2.4}$$

where $\eta \in [0,1)$. Then T has unique fixed point, provided the space Ω is complete.

Proof. Choose an arbitrary element $\zeta_0 \in \Omega$ and construct a sequence $\{\zeta_n\}$ as $\zeta_{n+1} = T\zeta_n$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose $\zeta_n \neq \zeta_{n+1}$ for all $n \geq 0$, otherwise ζ_n is a fixed point of T for some n and the result holds. Firstly, we prove that the terms of the sequence $\{\zeta_n\}$ are distinct. On contrary, if $\zeta_n = \zeta_m$ for some $n > m$, then $\zeta_n = \zeta_{m+k}$ for some $k \geq 1$ that implies $\zeta_{m+1} = \zeta_{m+k+1}$. Thus, by using inequality (2.4), we get that

$$\begin{aligned} d^e_{v_g b}(\zeta_{m+1}, \zeta_m) &= d^e_{v_g b}(\zeta_{m+k+1}, \zeta_{m+k}) \\ &= d^e_{v_g b}(T\zeta_{m+k}, T\zeta_{m+k-1}) \\ &\leq \eta d^e_{v_g b}(\zeta_{m+k}, \zeta_{m+k-1}) \\ &\leq \eta^2 d^e_{v_g b}(\zeta_{m+k-1}, \zeta_{m+k-2}) \\ &\vdots \\ &\leq \eta^k d^e_{v_g b}(\zeta_{m+1}, \zeta_m) \\ &< d^e_{v_g b}(\zeta_{m+1}, \zeta_m). \end{aligned} \tag{2.5}$$

Therefore, our supposition that $\zeta_n = \zeta_m$ for some $n > m$ is not true and hence the terms of the sequence $\{\zeta_n\}$ are distinct. Moreover, inequality (2.4) gives that $d^e_{v_g b}(\zeta_m, \zeta_n) \leq \eta d^e_{v_g b}(\zeta_{m-1}, \zeta_{n-1})$ for all m, n. Due to Lemma 8, it follows that the sequence $\{\zeta_n\}$ is Cauchy in Ω and it converges to some point say $\zeta \in \Omega$ as the space $(\Omega, d^e_{v_g b})$ is complete. Now, we claim that the point $\zeta \in \Omega$ is fixed point of map T. As the sequence $\{\zeta_n\}$ converges to ζ and $\zeta_n = T^n\zeta_0$, then $T^n\zeta_0 \to \zeta$. Again, due to inequality (8), we have

$$\begin{aligned} d^e_{v_g b}(T^{n+1}\zeta_0, T\zeta) &\leq \eta d^e_{v_g b}(T^n\zeta_0, \zeta) \\ &= \eta d^e_{v_g b}(\zeta_n, \zeta). \end{aligned}$$

Taking $n \to \infty$, we get that $d^e_{v_g b}(T^{n+1}\zeta_0, T\zeta) \to 0$ and consequently $\zeta_{n+1} \to T\zeta_0$. Therefore, on account of Lemma 9, it follows that $T\zeta = \zeta$. Thus, ζ is a fixed point of the map T. If ζ_1, ζ_2 are two fixed point of T in the space Ω. Then, in lieu of inequality (8), we obtain

$$d^e_{v_g b}(\zeta_1, \zeta_2) \leq \eta d^e_{v_g b}(\zeta_1, \zeta_2) < d^e_{v_g b}(\zeta_1, \zeta_2).$$

This is not true and thus the fixed point of T is unique. □

Remark 11.

If we take $\phi(t) = st$, where $s \geq 1$, then we obtain the Banach contraction principle in $b_v(s)$-metric space that means we obtain Theorem 2.1 of [5] and Theorem 9 of [9] as a special case of Theorem 10.

If we take $v = 2$, then we obtain the Banach fixed point theorem for extended rectangular b-metric space and hence Theorem 2.1 of [10] is a particular case of Theorem 10.

3. Discussion

The space we introduced here is generalizing not only the metric space but also the various generalized versions of metric space existing in the literature, for example, b-metric space, rectangular metric space, rectangular b-metric spaces etc. Therefore, the fixed point result presented here is important in the sense that it extends the existing Banach contraction principle in extended rectangular b-metric spaces and $b_v(s)$-metric spaces.

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