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Dynamics of Ratio-Dependent Eco-Epidemiological Model with Prey Refuge and Prey Harvesting

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Abstract: In this paper, an eco-epidemiological model of prey refuge and prey harvesting in infected prey populations is discussed. The predator consumes susceptible and infected prey at different rates in the form of a ratio-dependent type of interaction. The existence, positive invariance, and boundedness of the system are addressed. We have also established the stability of equilibrium points. The occurrence of Hopf-bifurcation is examined by analyzing the distribution of eigenvalues at the interior equilibrium point. Finally, to support the primary analytical findings, some numerical simulations were also given.

Keywords: Eco-epidemiological model, Ratio-dependent, Prey refuge and harvesting, Stability, Hopf-bifurcation

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1. Introduction

Eco-epidemiological models are used to determine the interaction between predator and prey with infection in one population or in a susceptible and infected prey population. Mathematical models have become major tools in analyzing the spread and control of diseases. Lotka [1] and Volterra [2] Predator-prey models, in the form of a coupled system of non-linear differential equations, can be considered the first breakthrough in modern mathematical ecology. In these models, the main concern is to study equilibrium points, their stability analysis, periodic solutions, bifurcations, chaotic behavior, etc. A mathematical representation of interactions between predator-prey, called "functional response," is one of the key components of predator-prey population modelling. There are several different types of functional responses, including the Holling type I-III [3], [4]; Hassell-Varley type; Beddington-DeAngelis type; Crowley-Martin type; and the recently well-known ratio-dependent type by Arditi and Ginzburg [5]. The prey's predation rate is assumed to be a function of the number of prey a predator consumes per unit of time. Many authors started to examine the predator-prey model with infection in either the population of prey or the predator or both populations [6]. The two types of disease in the predator population model with the linear functional response as well as the type II of Holling function were examined by Kadhim and Azhar [7]. The non-linear investigation of the predator-prey model with distinct effects was investigated in [8]. In [9] discussed the global and local stability analyses and also investigated the analysis of bifurcation for the ratio-dependent intra-guild predation model. The prey-predator food web model with the type II of Holling function was examined by Magudeeswaran et al. [10]. Commercial exploitation of ecological resources to meet the growing needs of society has been a topic of much concern for ecologists, bio-economists and natural resource managers. Harvesting is commonly practised in fisheries, forestry and in wildlife management. These investigations revealed very rich and interesting dynamics such as stability of equilibria, existence of Hopf bifurcation, limit cycles, homoclinic loops, Bogdanov-Takens bifurcation, and even catastrophe. In eco-epidemiology, we study predator-prey models along with disease dynamics. We are interested in investigating the dynamics of the predator-prey model using this functional response. A lot of research has been done on the dynamic behavior of ratio-dependent eco-epidemiological models. Several authors investigated the non-linear behavior of the infected predator prey model with various functional responses [11]. To the best of our knowledge, no researchers have studied the three species of prey-predator models that incorporate species interaction, as in the function of ratio-dependent disease in the prey populations. In this paper, we investigate the dynamics of a ratio-dependent eco-epidemiological model with prey refuge and prey harvesting. Here, we have studied the boundedness, positivity, local and global stability of the non-equilibrium points of this system.

2. The Mathematical Model Formation

The following set of nonlinear differential equations is used in an eco-epidemiological model with a refuge and prey harvesting.

$$\left.\begin{aligned} \frac{dU}{dT} &= RU\left(1-\frac{U+V}{K}\right)-\frac{\alpha_1 UV}{U+V}-H_1E_1U, \\ \frac{dV}{dT} &= \frac{c\alpha_1 UV}{U+V}-d_1V-\frac{b_1(1-m)VW}{a_1W+(1-m)V}-H_2E_2V, \\ \frac{dW}{dT} &= -d_2W+\frac{cb_1(1-m)VW}{a_1W+(1-m)V}, \end{aligned}\right\} \tag{1}$$

with non-negative conditions $U(0) \geq 0, V(0) \geq 0$ and $W(0) \geq 0$.

Table 1. Environmental representation of the system

Parameters	Environmental representation
U,V,W	Susceptible Prey, infected prey, predator
R	Prey growth rate
K	Carrying capacity of the environment
a_1	The constant for half-saturation
α_1	Rate of predation on susceptible prey
b_1	Predation rate of prey with infected
c	Prey to preadator conversion rate
d_1, d_2	Infected prey death rate, Predator population death rate
m	Refuge of prey
H_1, H_2	Prey's catchability coefficient
E	Harvesting effort

It is convient to scale the variable to minimize the amount of variables of the system (1) as $u = \frac{U}{K}$, $v = \frac{V}{K}$, $w = \frac{a_1W}{K}$ and to consider the non-dimensional time $t = RT$ the transforms leads to non-dimensional system. Now the system (1) becomes,

$$\left.\begin{aligned} \frac{du}{dt} &= u(1-u-v)-\frac{\alpha uv}{u+v}-h_1u, u(0) \geq 0, \\ \frac{dv}{dt} &= \frac{c\alpha uv}{u+v}-dv-\frac{b(1-m)vw}{w+(1-m)v}-h_2v, v(0) \geq 0, \\ \frac{dw}{dt} &= -\delta w+\frac{cb(1-m)vw}{w+(1-m)v}, w(0) \geq 0, \end{aligned}\right\} \tag{2}$$

where $\alpha = \frac{\alpha_1}{R}, b = \frac{b_1}{R}, h_1 = \frac{H_1E_1}{R}, d = \frac{d_1}{R}, h_2 = \frac{H_2E_2}{R}, \delta = \frac{d_2}{R}$.

3. POSITIVITY AND BOUNDEDNESS OF THE SOLUTION

3.1. Positive Invariance

Let $X \equiv (u(t), v(t), w(t))^T$ and $\mathcal{E}(X) = (\mathcal{E}_1(X), \mathcal{E}_2(X), \mathcal{E}_3(X))^T$, where

$$\mathcal{E}_1(X) = u(1-u-v)-\frac{\alpha uv}{u+v}-h_1u, \mathcal{E}_2(X) = \frac{c\alpha uv}{u+v}-dv-\frac{b(1-m)vw}{w+(1-m)v}-h_2v,$$

$$\mathcal{E}_3(X) = -\delta w+\frac{cb(1-m)vw}{w+(1-m)v}.$$

Then, the system (2) can be written as $\frac{dX}{dt} = \mathcal{E}(X)$ where $\mathcal{E} : C_{+}\rightarrow \mathcal{R}^3_+$ with $X(0) = X_0 \in \mathcal{R}^3_+$. Here, $\mathcal{E}_j \in \mathcal{C}^{\infty}(\mathcal{R})$ for $j = 1, 2, 3$. Thus, the function $\mathcal{E}$ is Lipschitzian and continuous on $\mathcal{R}^3_+$. The system (2) has positive initial conditions it can shown that these solutions exist. Hence, (2) is an invariant in the region $\mathcal{R}^3_+$.

3.2. Boundedness

Theorem 1. *Each and every one of the model (2) solutions is non-negative and uniformly bounded.*

Proof: Let $(u(t), v(t), w(t))$ be any solutions of the model (2) with positive initial conditions. Since,

$$\tfrac{du}{dt} \leq u(1-u).$$

From the above inequality we have, $\limsup_{t\to\infty} u(t) \leq 1$. Let $\eta = u + v + w$.

$$\begin{aligned}
\frac{d\eta}{dt} =& u(1-u) - uv - \frac{(1-c)\alpha uv}{u+v} - h_1 u - dv - \frac{(1-c)b(1-m)vw}{w+(1-m)v} - h_2 v - \delta w, \\
\leq& u(1-u) - h_1 u - (d+h_2)v - \delta w \text{ (since } c < 1), \\
\leq& \frac{r}{4} - h_1 u - (d+h_2)v - \delta w \text{ (since } Max\ \{u(1-u)\} = \frac{r}{4}), \\
\leq& \frac{r}{4} - \gamma\eta, \text{where } \gamma = min\ \{h_1, d+h_2, \delta\}.
\end{aligned}$$

Hence, we have $\frac{d\eta}{dt} + \gamma\eta \leq \frac{r}{4}$. Using the theorem of differential inequality, we obtain

$$0 < \eta \leq \tfrac{r}{4\gamma}\ (1 - exp^{-\gamma t}) + \eta\ (u_0, v_o, w_0)\ exp^{-\gamma t}.$$

For $t\rightarrow\infty$, we have $0 < \eta \leq \frac{r}{4\gamma}$. Hence, all the solutions of the system (2) starting in $\mathcal{R}^3_+$ for any $\in > 0$ are confined in the region $\Omega = \{(u, v, w) \in \mathcal{R}^3_+ : u + v + w \leq \frac{r}{4\gamma} + \in\}$. ☐

4. Existence of Equilibria

The model (2) has the below equilibrium points:

1. E_1 (1, 0, 0) is the boundary equilibrium point.
2. E_2 $(\bar{u}, \bar{v}, 0)$ is the predator-free equilibrium point, exists if $\alpha + h_1 > 1$ and $\frac{d+h_2}{\alpha} < c < \frac{d+h_2}{\alpha+h_1-1}$. When condition holds, $\bar{u}, \bar{v}$ are given by

   $$\bar{u} = \tfrac{(d+h_2)}{c^2\alpha}(c(1-h_1-\alpha)+d+h_2), \bar{v} = \tfrac{(c\alpha-(d+h_2))}{c^2\alpha}(c(1-h_1-\alpha)+d+h_2).$$

3. The endemic equilibrium point $E^*(u^*, v^*, w^*)$, exists if the following two conditions are satisfied: $(i)\frac{A}{\alpha+h_1} < bc^2 < \frac{A}{\alpha+h_1-1}$ and $(ii)cb > \delta$, where $A = (d+h_2)cb + b(1-m)(cb-\delta)$. Furthermore, u^*, v^*, w^* are given by

   $$\begin{aligned}
   u^* =& \frac{A}{c^4\alpha^2b^2}(c^2\alpha b - \alpha(c^2b(\alpha+h_1) - A)),\ v^* = \frac{c^2\alpha b - A}{c^4\alpha^2b^2}(c^2\alpha b - \alpha(c^2b(\alpha+h_1) - A)), \\
   w^* =& \frac{(A-(d+h_2)cb)(c^2\alpha b - A)}{b\delta c^4\alpha^2b^2}(c^2\alpha b - \alpha(c^2b(\alpha+h_1) - A)).
   \end{aligned}$$

   The constant of refuge should lies in the interval

   $$1 - \tfrac{c(c(\alpha+h_1)-(d+h_2))}{cb-\delta} < m < 1 - \tfrac{c(c(\alpha+h_1-1)-(d+h_2))}{cb-\delta}. \tag{3}$$

5. Local stability analysis

We want to calculate the Jacobian matrix for local stability analysis around different equilibrium points. The Jacobian matrix at an arbitrary point (u, v, w) is given by

$$J(E) = \begin{bmatrix} 1 - 2u - v - \frac{\alpha v^2}{(u+v)^2} - h_1 & -u - \frac{\alpha v^2}{(u+v)^2} & 0 \\ \frac{c\alpha v^2}{(u+v)^2} & \frac{c\alpha u^2}{(u+v)^2} - d - h_2 - \frac{b(1-m)w^2}{(w+(1-m)v)^2} & -\frac{b(1-m)^2 v^2}{(w+(1-m)v)^2} \\ 0 & \frac{bc(1-m)w^2}{(w+(1-m)v)^2} & -\delta + \frac{bc(1-m)^2 v^2}{(w+(1-m)v)^2} \end{bmatrix}.$$

Theorem 2. *For model (2), we have*

1. *If $\alpha + h_1 > 1$ and $\frac{d+h_2}{\alpha} < c < \frac{d+h_2}{\alpha+h_1-1}$, a system of (2) has an equilibrium point E_2 is locally asymptotically stable, .*
2. *$E^*(u^*, v^*, w^*)$ is the interior equilibrium point is locally asymptotically stable, if $A > 0, C > 0, AB - C > 0$.*

Proof: 1. At an equilibrium point E_2, the model (2) Jacobian matrix is given by

$$J(E_2) = \begin{bmatrix} -\bar{u} + \frac{\alpha\bar{u}\bar{v}}{(\bar{u}+\bar{v})^2} & -\bar{u} - \frac{\alpha\bar{u}^2}{(\bar{u}+\bar{v})^2} & 0 \\ \frac{c\alpha\bar{v}^2}{(\bar{u}+\bar{v})^2} & -\frac{c\alpha\bar{u}\bar{v}}{(\bar{u}+\bar{v})^2} & -b \\ 0 & 0 & bc - \delta \end{bmatrix}.$$

The characteristic equation of $J(E_2)$ is $(\lambda^2 + M\lambda + N)(\lambda - bc + \delta) = 0$. Where,

$$M = \bar{u} + \frac{(c-1)\alpha\bar{u}\bar{v}}{(\bar{u}+\bar{v})^2} \text{ and } N = \frac{c\alpha\bar{u}\bar{v}}{\bar{u}+\bar{v}} > 0.$$

The eigenvalues of the above Jacobian matrix at E_2 are $\lambda_{1,2} = -\frac{-M \pm \sqrt{M^2-4N}}{2}$ and $\lambda_3 = bc - \delta$.

2. The Jacobian matrix at E^* (u^*, v^*, w^*) is

$$J(E^*) = \begin{bmatrix} -u^* + \frac{\alpha u^* v^*}{(u^*+v^*)^2} & -u^* - \frac{\alpha u^{*2}}{(u^*+v^*)^2} & 0 \\ \frac{c\alpha v^{*2}}{(u^*+v^*)^2} & -\frac{c\alpha u^* v^*}{(u^*+v^*)^2} + \frac{b(1-m)^2 v^* w^*}{(w^*+(1-m)v^*)^2} & -\frac{b(1-m)^2 v^{*2}}{(w^*+(1-m)v^*)^2} \\ 0 & \frac{bc(1-m)w^{*2}}{(w^*+(1-m)v^*)^2} & -\frac{bc(1-m)v^* w^*}{(w^*+(1-m)v^*)^2} \end{bmatrix}.$$

The cubic characteristic equation of $J(E^*)$ is $\lambda^3 + A\lambda^2 + B\lambda + C = 0$, where

$$\mathcal{A} = -b_{11} - b_{22} - b_{33} > 0, \mathcal{B} = b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{23}b_{32} - b_{12}b_{21},$$
$$\mathcal{C} = -det[J(E^*)] = b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} - b_{11}b_{22}b_{33} > 0.$$

By the criterion Routh-Hurwitz, E^* is locally asymptotically stable, if $\mathcal{A} > 0, \mathcal{B} > 0$, and $\mathcal{AB} - \mathcal{C} > 0$.

□

6. Global Stability

In this section, we studied the global stability of the system (2) around the positive equilibrium $E^*(u^*, v^*, w^*)$.

Theorem 3. *The point of interior equilibrium is globally asymptotically stable if $\alpha(A - bc^2\alpha) + bc^2\alpha > 0$ and $\frac{c\alpha u^*}{(u+v)(u^*+v^*)} - \frac{b(1-m)^2 w^*}{(w+(1-m)v)(w^*+(1-m)v^*)} > 0$, for all $(u, v, w) \in \mathcal{R}^3_+$, then E^* is globally asymptotically stable.*

Proof. A function of Lyapunov form

$$L_1(u, v, w) = L_2(u - u^* - u^* ln\frac{u}{u^*}) + (v - v^* - v^* ln\frac{v}{v^*}) + L_3(w - w^* - w^* ln\frac{w}{w^*}),$$

where L_2, L_3 are positive constant. Differentiating L_1 with respect to t along the solution of (2), a little algebraic manipulation yields

$$
\begin{aligned}
\frac{dL_1}{dt} =& - L_2\{1 - \frac{\alpha v^*}{(u+v)(u^*+v^*)} - h_1\}(u-u^*)^2 - L_2(u-u^*)(v-v^*) \\
&+ \frac{(c\alpha v^* - A\alpha u^*)}{(u+v)(u^*+v^*)}(u-u^*)(v-v^*) \\
&- \{\frac{c\alpha u^*}{(u+v)(u^*+v^*)} - \frac{b(1-m)^2 w^*}{(w+(1-m)v)(w^*+(1-m)v^*)} - h_2\}(v-v^*)^2 \\
&- \frac{Bcb(1-m)v^*}{(w+(1-m)v)(w^*+(1-m)v^*)}(w-w^*)^2 \\
&+ (1-m)\frac{Bcbw^* - b(1-m)v^*}{(w+(1-m)v)(w^*+(1-m)v^*)}(v-v^*)(w-w^*).
\end{aligned}
$$

We choose $L_2 = \frac{cv^*}{u^*}$ and $L_3 = \frac{(1-m)v^*}{cw^*}$.

$$
\begin{aligned}
\frac{dL_1}{dt} =& - \frac{cv^*}{u^*}\{1 - \frac{\alpha v^*}{(u+v)(u^*+v^*)} - h_1\}(u-u^*)^2 \\
&- \frac{b(1-m)^2 v^* 2}{w^*(w+(1-m)v)(w^*+(1-m)v^*)}\}(v-v^*)^2 - \frac{cv^*}{u^*}(u-u^*)(v-v^*) \\
&- \{\frac{c\alpha u^* h_2}{(u+v)(u^*+v^*)} - \frac{b(1-m)^2 w^*}{(w+(1-m)v)(w^*+(1-m)v^*)}\}(v-v^*)^2.
\end{aligned}
$$

Now the condition $\alpha(A - bc^2\alpha) + bc^2\alpha > 0$ implies that $1 - \frac{\alpha v^*}{(u+v)(u^*+v^*)} > 0$. Thus, E^* is globally asymptotically stable. □

7. Hopf-bifurcation analysis

We examine the model's bifurcation in this section based on the rate of refuge. The theorem below demonstrates that for the bifurcating parameter m, a given Hopf-bifurcation exists.

Theorem 4. *If the bifurcation parameter m exceeds a significant value, the model (2) undergoes Hopf-bifurcation. The occurrence of Hopf-bifurcation conditions at $m = m^*$ as follows,*

1. $\mathcal{L}(m^*)\mathcal{M}(m^*) - \mathcal{N}(m^*) = 0$,
2. $\frac{d}{dm}(Re(\lambda(m)))|_{m=m^*} \neq 0$, *where λ is the zero of the characteristic equation corresponds to the positive equilibrium point.*

Proof. For $m = m^*$, let the characteristic equation (3) implies that

$$
(\lambda^2(m^*) + \mathcal{M}(m^*))(\lambda(m^*) + \mathcal{L}(m^*)) = 0. \tag{4}
$$

which implies that $\pm i\sqrt{\mathcal{M}(m^*)}$ and $-\mathcal{L}(m^*)$ are the roots of the above equation (4). We need to satisfy the following transversality condition to demonstrate that the Hopf-bifurcation at the point $m^* = m$. $\frac{d}{df}(Re(\lambda(m)))|_{m=m^*} \neq 0$. For all m, the general roots of the form $\lambda_{1,2}(m) = r(m) \pm is(m)$, and $\lambda_3(m) = -A_1(m)$. Now, we check the condition $\frac{d}{dm}(Re(\lambda_j(m)))|_{m=m^*} \neq 0, j = 1, 2$. Let, $\lambda_1(m)= r(m) + is(m)$ in (4), we get $\zeta_1(m) + i\zeta_2(m) = 0$. Where,

$$
\begin{aligned}
\zeta_1(m) =& r^3(m) + r^2(m)\mathcal{L}(m) - 3r(m)s^2(m) - s^2(m)\mathcal{L}(m) + r(m)\mathcal{M}(m) + \mathcal{L}(m)\mathcal{M}(m), \\
\zeta_2(m) =& 3r^2(m)s(m) + 2r(m)s(m)\mathcal{L}(m) - s^3(m) + s(m)\mathcal{M}(m).
\end{aligned}
$$

In order to fulfill the equation (4), we must have $\zeta_1(m) = 0$ and $\zeta_2(m) = 0$, then differentiating ζ_1 and ζ_2 with respect to m. We have

$$\frac{d\zeta_1}{dm} = \phi_1(m)r'(m) - \phi_2(m)s'(m) + \phi_3(m) = 0, \tag{5}$$

$$\frac{d\zeta_2}{dm} = \phi_2(m)r'(m) + \phi_1(m)s'(m) + \phi_4(m) = 0, \tag{6}$$

where,

$$\begin{aligned}
\phi_1(m) &= 3r^2(m) + 2r(m)\mathcal{L}(m) - 3s^2(m) + \mathcal{M}(m), \phi_2(m) = 6r(m)s(m) + 2s(m)\mathcal{L}(m),\\
\phi_3(m) &= r^2(m)\mathcal{L}'(m) - s^2(m)\mathcal{L}'(m) + \mathcal{N}'(m) + \mathcal{M}'(m)r(m),\\
\phi_4(m) &= 2r(m)s(m)\mathcal{L}'(m) + s(m)\mathcal{M}'(m).
\end{aligned}$$

On multiplying (5) and (6) by $\phi_1(m)$ and $\phi_2(m)$ respectively,

$$r'(m) = -\frac{\phi_1(m)\phi_3(m) + \phi_2(m)\phi_4(m)}{\phi_1^2(m) + \phi_2^2(m)}. \tag{7}$$

Substituting $r(m) = 0$ and $s(m) = \sqrt{\mathcal{M}(m)}$ at $m = m^*$ on $\phi_1(m), \phi_2(m), \phi_3(m)$ and $\phi_4(m)$ we obtain

$$\begin{aligned}
&\phi_1(m^*) = -2\mathcal{M}(m^*), \phi_2(m^*) = 2\sqrt{\mathcal{M}(m^*)}\mathcal{L}(m^*),\\
&\phi_3(m^*) = -\mathcal{M}(m^*)\mathcal{L}'(m^*) + \mathcal{N}'(m^*), \phi_4(m^*) = \sqrt{\mathcal{M}(m^*)}\mathcal{M}'(m^*).
\end{aligned}$$

The equation (7), implies

$$r'(m^*) = \frac{\mathcal{N}'(m^*) - (\mathcal{L}(m^*)\mathcal{M}'(m^*) + \mathcal{M}(m^*)\mathcal{L}'(m^*))}{2(\mathcal{M}^2(m^*) + \mathcal{L}^2(m^*))}, \tag{8}$$

If $\mathcal{N}'(m^*) - (\mathcal{L}(m^*)\mathcal{M}'(m^*) + \mathcal{M}(m^*)\mathcal{L}'(m^*)) \neq 0$, which implies that $\frac{d}{dm}(Re(\lambda_j(m)))|_{m=m^*} = r'(m^*) \neq 0$. $j = 1, 2$, and $\lambda_3(m^*) = -\mathcal{L}(m^*) \neq 0$. The condition $\mathcal{N}'(m^*) - (\mathcal{L}(m^*)\mathcal{M}'(m^*) + \mathcal{M}(m^*)\mathcal{L}'(m^*)) \neq 0$, is confirmed that the transversality criteria hold, thus the model (2) undergoes the Hopf-bifurcation at $m = m^*$. □

8. Numerical Simulations

In this part, we show some few numerical simulations on the system (2) are performed in this section to validate the theoretical conclusions. The system parameters are (2) as $\alpha = 1.2, b = 1.8, c = 0.85, d = 0.2, \delta = 0.8, h_1 = 0.2, h_2 = 0.15$, and $(u(0), v(0), w(0)) = (0.5, 0.5, 0.5)$. Also, we choose $m = 0.6$ since it meets the inequality (3). Hence, $E^*(u^*, v^*, w^*) = (0.2957, 0.2089, 0.1250)$ is locally asymptotically stable and the criteria of Theorem 2 are satisfied as $A = 0.5706 > 0, AB - C = 0.0461 > 0$.

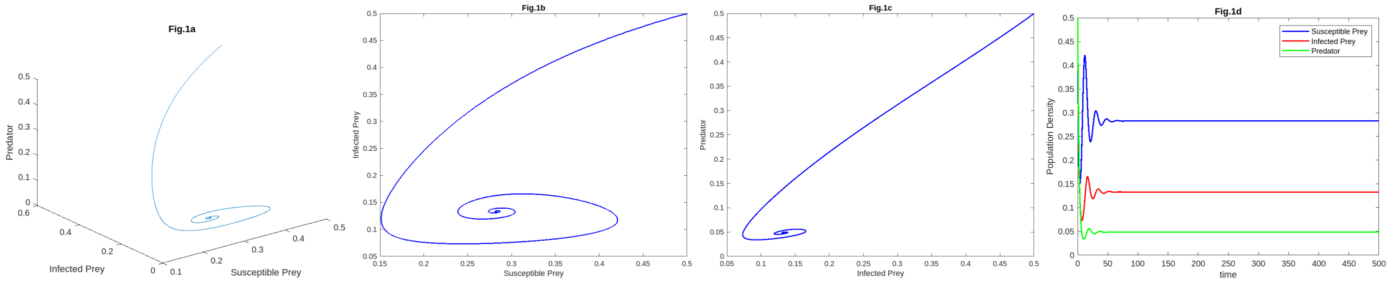

Figure 1. Here $u(0) = 0.5, v(0) = 0.5, w(0) = 0.5$ and $\alpha = 1.2, b = 1.8, c = 0.85, d = 0.2, \delta = 0.8, h_1 = 0.2, h_2 = 0.15, m = 0.6$. (a) Phase portrait of the system (2) at E^*(b) uv-plane (c) vw-plane (d) Time series of system (2).

9. Discussion

In this study, we examined an eco-epidemiological model that included a prey refuge, prey harvesting, and infection in the population of prey. The model was controlled by a modified logistic equation. A more accurate model is produced when a refuge and harvesting are included in the system (1). The biological control of a pest may benefit from a refuge and harvesting; yet, expanding the refuge could result in higher prey numbers and population outbreaks. In addition, increasing the pace at which susceptible prey is harvested causes an increase in the number of infected prey while decreasing the population of predators and susceptible prey. Theorem 1 establishes that the non-dimensionalized system (refeqn2) is uniformly bounded, which suggests that the system exhibits good biological behaviour. The underlying premise that most epidemic models we see in nature correspond to stable equilibria is typically the one that theoretical epidemiologists follow in deterministic situations. The most significant equilibrium point from this perspective is $E*$ $(u*, v*, w*)$, which we have shown. Now we observe that the size of the susceptible prey (u) in the absence and presence of infected prey (v) is $\bar{u} = \frac{(d+h_2)}{c^2\alpha}(c(1-h_1-\alpha)+d+h_2)$ and $u^* = \frac{A}{c^4\alpha^2b^2}(c^2\alpha b - \alpha(c^2b(\alpha+h_1)-A))$, respectively, so that $u^* - \bar{u} = \frac{b(1-m)(bc-\delta)}{b^2c^4\alpha^2}[(c^2\alpha b - \alpha(c^2b(\alpha+h_1)-A)) + bc\alpha(d+h_2)] > 0$ whenever $bc > \delta$. This indicates that if the force of infection is high, then predator attack causes enhancement of the susceptible prey. All our important mathematical findings are numerically verified, and a graphical representation of a variety of solutions to the system (2) is depicted using MATLAB. Finally, our model can be generalized in obvious ways to food chains and competitive systems.

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