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Dynamical Analysis of a Fractional Order Prey-Predator Model in Crowley-Martin Functional Response with Prey Harvesting †

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Abstract: In this paper, we investiage a fractional-order predator-prey model incorporating prey harvesting. In a non-delayed model, the functional response of Crowley-Martin has been studied. We first prove the existence, uniqueness, non-negativity and boundedness of the solutions for the proposed model. Furthermore, analyze the existence of various equilibrium points to examine the local asymptotically stable properties, and use the suitable Lyapunov function to study the globally asymptotic stability. Finally, some numerical simulations are verified for the analytic results.

Keywords: Crowley-Martin functional response; existence and uniqueness; stability

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1. Introduction

Fractional Calculus (FC) is a general field that attempts to understand real-world phenomena.A non-integer sequence is modeled with derivatives and is a field of differentiation and integrations are performed with non-integer order derivatives. The memory impact and conserved relevant physical properties are the benefits of fractional derivatives.The predator prey models developed by Lotka and Voltera are considered early developments in contemporary mathematical eology in coupled system of non-linear differential equations [1,2]. Since Kermack-Mckendrick's pioneering work on SIRS, epidemiological models have attarcted much interest from researchers [3]. In general, there are two main types of mathematical models: ecological and epidemiological models. The relations between populations of a certain community are explored in ecological models [4]. Epidemiology models are studies of how illnesses spread between humans and animals. This article's major objective is to investigate how infection affects prey during prey harvesting in a predator-prey system. Here, we examined the local and global stabilities of the equilibrium points of this system, as well as the boundedness and positivity of the solution [5,6]. An eco-epidemiological predator-prey system with disease affecting only prey species and harvesting of both susceptible and infected prey has been taken into consideration by Bhattacharya et al. [7,8]. Agnihotri and Gakkhar studied a prey-predator system with disease affecting both species and only the prey species being harvested [9,10]. The SIRS-type models are mathematically similar to models of the s-called geometric Brownian motion (GBM): both predict an exponential growth, of infected individuals or of the value of the process. The state of dead individuals in the SIRS models is akin to the so-called resetting in the GBM model [11,12]. The very relevant model of GBM with resetting was considered recently in refs.Ecologists, economists, and those involved in natural resource management have been interested in the studies on harvesting in predator-prey systems for some time. Few people have specifically included a harvested parameter in a predator-prey-parasite model and analysed the system's response to it [13]. In this article, we examine the function of harvesting in an eco-epidemiological system where susceptible and diseased prey are harvested together [14,15]. This study applies the Caputo fractional derivative and the harvesting rate to the predator-prey model. This author's major objective is to investigate the effects of prey infection and prey harvesting in a predator-prey system. Here, we examined the boundedness, positivity, local and global stabilities, and stability of the system's equilibrium points.

2. Model Formulation

The model explains how the diseased prey system interact with harvesting, which results in the following set of equations. The dynamical prey and predator mathematical model was investigated using the proposed model,

$$\left.\begin{aligned} \frac{dL}{dT} &= r_1 L(1-\frac{L+M}{K}) - \lambda ML - \frac{\alpha_1 LN}{(1+\zeta L)(1+\eta N)} - H_1E_1L, \\ \frac{dM}{dT} &= \lambda ML - d_1M - \frac{b_1MN}{a_1+M} - H_2E_2M, \\ \frac{dN}{dT} &= -d_2M + \frac{cb_1MN}{a_1+M} + \frac{c\alpha_1 LN}{(1+\zeta L)(1+\eta N)}. \end{aligned}\right\} \tag{1}$$

Table 1. Biological representation of the system (1) parameters.

Parameters	Biological Representation
L	Susceptible Prey
M	Infected Prey
N	Predator
r_1	Intrinssic Prey growth rate
K	Carrying capacity of the environment
α_1	Predation rate of Susceptible Prey
b_1	Predation rate of Infected Prey
a_1	Half-saturation constant
c	Conversion coefficient from the prey to predator
d_1	Infected Prey death rate
d_2	Predator Population death rate
λ	Infection Rate

The variables $l = \frac{L}{K}$, $m = \frac{M}{K}$, $n = \frac{N}{K}$ and the dimension time $t = \lambda KT$ can be changed in order to reduce the number of parameters in the system (1). We apply the following transformations

$$s_1 = \frac{r_1}{\lambda k}, s_2 = \frac{\alpha_1}{\lambda K}, s_3 = \frac{a_1}{k}, s_4 = \frac{\alpha_1}{\lambda K}, s_5 = \frac{b_1}{\lambda k}, s_6 = \frac{d_2}{\lambda k}.$$

According to the above transformations the Equation (1) can be rewritten in the following non-dimensional form

$$\left.\begin{aligned} \frac{dl}{dt} &= s_1 l(1-l-m) - lm - \frac{s_2 ln}{(1+\zeta l)(1+\eta n)} - \theta_1 l, \\ \frac{dm}{dt} &= lm - s_{4m} - \frac{s_5 mn}{s_3+m} - \theta_2 m, \\ \frac{dn}{dt} &= -s_6 n + \frac{cs_5mn}{s_3+m} + \frac{cs_2 ln}{(1+\zeta l)(1+\eta n)}. \end{aligned}\right\} \tag{2}$$

In the system,we have taken fractional-order derivative α to model (2) with restore the fractional-order Caputo derivative. Then, the model (2) is take into the following form

$$\left.\begin{aligned}
\frac{d^\alpha l}{dt^\alpha} &= s_1 l(1-l-m) - lm - \frac{s_2 ln}{(1+\zeta l)(1+\eta n)} - \theta_1 l,\\
\frac{d^\alpha m}{dt^\alpha} &= lm - s_4 m - \frac{s_5 mn}{s_3+m} - \theta_2 m,\\
\frac{d^\alpha n}{dt^\alpha} &= -s_6 n + \frac{cs_5 mn}{s_3+m} + \frac{cs_2 ln}{(1+\zeta l)(1+\eta n)}.
\end{aligned}\right\} \tag{3}$$

subject to the initial conditions $l(0) \geq 0$, $m(0) \geq 0$, $n(0) \geq 0$.

3. Existence and Uniqueness of the Solutions

In this Section, boundedness of solution of the system (3) has been examined. The fractional-order system as follows:

$$\frac{d^\alpha X(t)}{dt^\alpha} = f(t, X(t)), \quad \alpha \in (0,1].$$

Theorem 1. *The fractional order system (3) has a unique solution, for the non-negative initial conditions.*

Proof. A sufficient condition for the solutions of system (3) in the region $\chi \times (0,T]$ where,

$$\chi = \{(l,m,n) \in R^3 : max(|l|,|m|,|n|) \leq \eta\}.$$

Now, let us define a mapping $V(X) = (V_1(X), V_2(X), V_3(X))$ where

$$\begin{aligned}
V_1(X) &= s_1 l(1-l-m) - lm - \frac{s_2 ln}{(1+\zeta l)(1+\eta n)} - \theta_1 l,\\
V_2(X) &= lm - s_4 m - \frac{s_5 mn}{s_3+m} - \theta_2 m,\\
V_3(X) &= -s_6 n + \frac{cs_5 mn}{s_3+m} + \frac{cs_2 ln}{(1+\zeta l)(1+\eta n)}.
\end{aligned}$$

$$\begin{aligned}
||V(X) - V(\bar{X})|| &= |V_1(X) - V_1(\bar{X})| + |V_2(X) - V_2(\bar{X})| + |V_3(X) - V_3(\bar{X})|\\
&= |s_1 l(1-l-m) - lm - \frac{s_2 ln}{(1+\zeta l)(1+\eta n)} - \theta_1 l - s_1\bar{l}(1-\bar{l}-\bar{m}) + \bar{l}\bar{m}\\
&+ \frac{s_2\bar{l}\bar{m}}{(1+\zeta\bar{l})(1+\eta\bar{n})} + \theta_1\bar{l}| + |lm - s_4 m - \frac{s_5 mn}{s_3+m} - \theta_2 m - \bar{l}\bar{m} + s_4\bar{m}\\
&+ \frac{s_5\bar{m}\bar{n}}{s_3+\bar{m}} + \theta_2\bar{m}| + | - s_6 n + \frac{cs_5 mn}{s_3+m} + \frac{cs_2 ln}{(1+\zeta l)(1+\eta n)} + s_6\bar{n} - \frac{cs_5\bar{m}\bar{n}}{s_3+\bar{m}}\\
&- \frac{cs_2\bar{l}\bar{n}}{(1+\zeta\bar{l})(1+\eta\bar{n})}|\\
&\leq \{s_1 + 2s_1\eta + (2+s_1\eta) + (1+c)\eta s_2 s_3 + \theta_1\}|l-\bar{l}|\\
&+ \{s_1\eta + (1+c)s_5\eta + s_4 + \theta_2\}|m-\bar{m}|\\
&+ \Big\{(1+c)s_2 s_3\eta + (1+c)s_2 s_3\eta^2 + (1+c)s_5\eta + s_6\Big\}|n-\bar{n}|\\
&\leq H\|X-\bar{X}\|.
\end{aligned}$$

where, $H = max\Big\{s_1 + 2s_1\eta + (2+s_1)\eta + \frac{(1+c)s_2 s_3\eta}{(s_2+\eta)^2} + \theta_1, (s_1+s_5+cs_5)\eta + s_4 + \theta_2,$ $\frac{(1+c)s_2 s_3\eta}{(s_3+\eta)^2} + \frac{(1+c)s_2 s_3\eta^2}{(s_3+\eta)^2} + (1+c)s_5\eta + s_6\Big\}$. Hence, the solution of the system (3) exist and unique. □

4. Equilibrium Points and Stability Analysis

In this section, the system (3) have the following possible equilibrium points:

(i) $E_0(0,0,0)$ is the trivial equilibirum point.

(ii) $E_1\left(\dfrac{s_1-\theta_1}{s_1},0,0\right)$ is the boundary equilibrium point.

(iii) $E_2(\bar{l},0,\bar{n})$ is the infected prey free equilibrium point, where $\bar{l}=\dfrac{s_6(1+\eta n)}{cs_2-s_6\zeta(1+\eta n)}$, and $\bar{n}=\dfrac{s_1(1-l)(1+\zeta l)}{s_2+\theta_1-\eta s_1(1-l)(1+\zeta l)}$.

(iv) $E_3(\hat{l},\hat{m},0)$ is the predator free equilibrium point, where $\hat{l}=s_4+\theta_2$ and $\hat{m}=\dfrac{s_1(1-s_4-\theta_2)-\theta_1}{s_1+1}$.

(v) The interior equilibrium point $E^*(l^*,m^*,n^*)$. Where,

$$m^*=\frac{s_3(s_3s_6+(s_6-cs_2)l^*)}{(cs_2l^*+(cs_5-s_6)(1+\zeta l^*)},n^*=\frac{s_3c(l^*+s_4-\theta_2)(1+\zeta l^*)}{(cs_2l^*+(cs_5-s_6)((1+\eta n^*)}$$

ans l^* is the unique positive root of the quadratic equation $Al^2+Bm+C=0$, Where $A=s_1(cs_2+cs_5-s_6)$, $B=(cs_5-s_6)(\theta_1-s_1+s_3s_1)+s_3c(\theta_1-s_1)+s_3(s_6+(s_6-cs_1)s_1)$, $C=-s_3((s_1-\theta_1)(cs_5-s_6)+(cs_2(s_4+\theta_2)-s_3s_4(1+s_1))$. Now, we want to calculate Jacobian matrix for local stability analysis around different equilibrium points.The Jacobian matrix at an arbitrary point (l,m,n) is given by

$$J(l,m,n)=\begin{pmatrix} n_{11} & n_{12} & n_{13}\\ n_{21} & n_{22} & n_{23}\\ n_{31} & n_{32} & n_{33}\end{pmatrix}.$$

where, $n_{11}=s_1(1-2l)-m(s_1+1)-\dfrac{s_2n}{(1+\zeta l)^2(1+\eta n)}-\theta_1, n_{12}=-l(s_1+1)$, $n_{13}=-\dfrac{s_2l}{(1+\zeta l)(1+\eta n)^2}$, $n_{21}=m$, $n_{22}=l-s_4-\theta_2-\dfrac{s_3s_5n}{(s_3+m)^2}$, $n_{23}=\dfrac{s_5m}{s_3+m}$, $n_{31}=\dfrac{cs_2n}{(1+\zeta l)^2(1+\eta n)}$, $n_{32}=\dfrac{s_3cs_5n}{(s_3+m)^2}$, $n_{33}=-s_6+\dfrac{cs_5m}{s_3+m}+\dfrac{s_2cl}{(1+\zeta l)(1+\eta n)^2}$.

Theorem 2. *The trivial equilibrium point $E_0(0,0,0)$ of a system (3) is stable if $s_1<\theta_1$, otherwise it is unstable.*

Proof. The Jacobian matrix of the system (3) at an equilibrium point E_0 is given by

$$J(E_0)=\begin{pmatrix} s_1-\theta_1 & 0 & 0\\ 0 & -s_4-\theta_2 & 0\\ 0 & 0 & -s_6\end{pmatrix}.$$

The eigenvalues are $\lambda_1=s_1-\theta_1$, $\lambda_2=-s_4-\theta_2$ and $\lambda_3=-s_6$. Hence, the trivial equilibrium point $E_0(0,0,0)$ is a stable if $s_1<\theta_1$, otherwise it is unstable. □

Theorem 3. *The infected free and predator free equilibrium point $E_1(\dfrac{s_1-\theta_1}{s_1},0,0)$ of a system (3) is stable if $cs_2(s_1-\theta_1)<s_6(s_1+\zeta(s_1-\theta_1))$.*

Proof. The Jacobian matrix of the system (3) at an equilibrium point E_1 is given by

$$J(E_1) = \begin{pmatrix} \theta_1 - s_1 & -\left(\frac{s_1 - h_1}{s_1}\right)(s_1+1) & \frac{-s_2(s_1-\theta_1)}{s_1s_3+(s_1-\theta_1)} \\ 0 & 1 - s_4 - \theta_2 - \frac{\theta_1}{s_1} & 0 \\ 0 & 0 & \frac{-s_2(s_1-\theta_1)}{s_1s_3+(s_1-\theta_1)-s_6} \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = \theta_1 - s_1$, $\lambda_2 = 1 - s_4 - \theta_2 - \frac{\theta_1}{s_1}$ and $\lambda_3 = \frac{-s_2(s_1-\theta_1)}{s_1s_3+(s_1-\theta_1)-s_6}$. Hence, the infected prey and predator free equilibrium point E_1 is stable. □

Theorem 4. *The disease free equilibrium point E_2 of a system (3) is locally asymptotically stable if $min\left\{\frac{s_3s_6}{c-s_6} - s_4, s_1\left(1 - \frac{2s_3s_6}{c-s_6}\right)\right\} < \theta_1$.*

Proof. $J(E_2) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$. where $d_{11} = -\theta_1 + s_1 - \frac{2s_3s_1s_6}{cs_2 - s_6} - \frac{(cs_2-s_6)^2\bar{n}}{s_3s_2c^2}$, $d_{12} = -\frac{s_3(1+s_1)s_6}{c-s_6}$, $d_{13} = -\frac{s_6}{c}$, $d_{21} = 0$, $d_{22} = -s_4 - \theta_2 + \frac{s_3s_6}{cs_2-s_6}$, $d_{23} = 0$, $d_{31} = \frac{((cs_2-s_6)^2\bar{n})}{s_3cs_2}$, $d_{32} = \frac{cs_5\bar{n}}{s_3}$, $d_{33} = 0$. Here, the characteristic equation of the above Jacobian matrix is

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0. \tag{4}$$

where, $P = -p_{11} - p_{22}$, $Q = -p_{31}p_{13} + p_{22}p_{11}$, $R = p_{13}p_{12}p_{31}$. According to Routh-Hurwitz criteria [16], $P > 0$, $R > 0$ and $PQ - R > 0$. Hence, E_2 is locally asymptotically stable. □

Theorem 5. *The equilibrium point E_3 of a system (3) is locally asymptotically stable if $s_6 > c(s_2 + s_5)$.*

Proof. The Jacobian matrix at E_3 is given by

$$J(E_3) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

where $m_{11} = -(s_4 + \theta_2)s_1$, $m_{12} = (-1 - s_1)\bar{l}$, $m_{13} = \frac{-s_2}{(1+\zeta l)^2)(1+\eta n)}$, $m_{21} = m$, $m_{22} = 0$, $m_{23} = \frac{s_5\bar{m}}{s_3+\bar{m}}$, $m_{31} = 0$, $m_{32} = 0$, $m_{33} = \frac{cs_2\bar{l}}{(1+\zeta l)(1+\eta n)^2} - s_6 + \frac{cs_5\bar{m}}{s_3+m}$. Here, the characteristic equation of the above Jacobian matrix is

$$\lambda^3 + E\lambda^2 + F\lambda + G = 0. \tag{5}$$

where, $E = -m_{11} - m_{33}$, $F = -m_{21}m_{12} + m_{33}m_{11}$, $G = m_{12}m_{21}m_{33}$. According to Routh-Hurwitz criteria [16], $E > 0$, $G > 0$ and $EF - G > 0$. Hence, E_3 is locally asymptotically stable. □

Theorem 6. *The endemic equilibrium point E^* of system (3) is locally asymptotically stable .*

Proof. The Jacobian matrix at E^* is given by

$$J(E^*) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

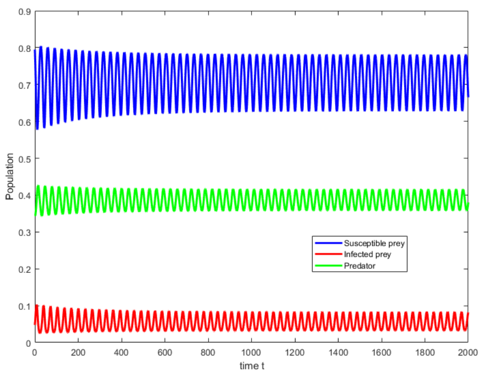
where, $g_{11} = s_1(1-2l^*) - m(s_1+1) - \frac{s_2 n}{(1+\zeta l^*)^2(1+\eta n^*)} - \theta_1$, $g_{12} = -l^*(s_1+1)$, $g_{13} = -\frac{s_2 l^*}{(1+\zeta l^*)(1+\eta n^*)^2}$, $g_{21} = m^*$, $g_{22} = l^* - s_4 - \theta_2 - \frac{s_3 s_5 n^*}{(s_3+m^*)^2}$, $g_{23} = \frac{s_5 m^*}{s_3+m^*}$, $g_{31} = \frac{c s_2 n^*}{(1+\zeta l^*)^2(1+\eta n^*)}$, $g_{32} = \frac{s_3 c s_5 n^*}{(s_3+m^*)^2}$, $g_{33} = -s_6 + \frac{c s_5 m^*}{s_3+m^*} + \frac{s_2 c l^*}{(1+\zeta l^*)(1+\eta n^*)^2}$.
Here, the characteristic equation of the above Jacobian matrix is

$$\lambda^3 + E\lambda^2 + F\lambda + G = 0. \quad (6)$$

where $E = -g_{11} - g_{33}$, $F = g_{21}g_{12} + g_{22}g_{11} - g_{13}g_{31} + g_{23}g_{32}$, $G = g_{13}(-g_{22}g_{31} + g_{21}g_{32}) + g_{23}(g_{12}g_{31} - g_{11}g_{32})$. According to Routh-Hurwitz criteria [16], $E > 0$, $G > 0$ and $EF - G > 0$. Hence, E^* is locally asymptotically stable. □

5. Numerical Analysis

In this section, we present some numerical simulation results for Caputo-sense fractional-order eco-epidemic models. To accomplish this, we use Diethelm et al.'s predictor-corrector approach to solve the defined model.the parameter values are chosen as $s_1 = 0.5$, $s_2 = 0.15$, $s_3 = 0.2$, $s_4 = 0.1$, $s_5 = 0.4$, $s_6 = 0.1$, $c = 0.5$, $\zeta = 0.5$, $\eta = 0.3$ and different values of $\alpha = 1$ and then the equilibrium point $E_4(0.794787, 0.0476298, 0.343099)$ is unstable (see Figure 1). Fixing the derivative α as a variable. Here we consider the derivative value as $\alpha = 0.92$, the effect of the predator harvesting θ_1 on the evolution of the three species and clearly influences the final size of the three populations are shown in Figure 2.



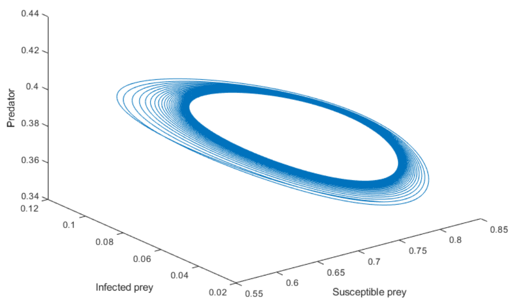


Figure 1. Unstable solution for the interior equilibrium point $\alpha = 1$.

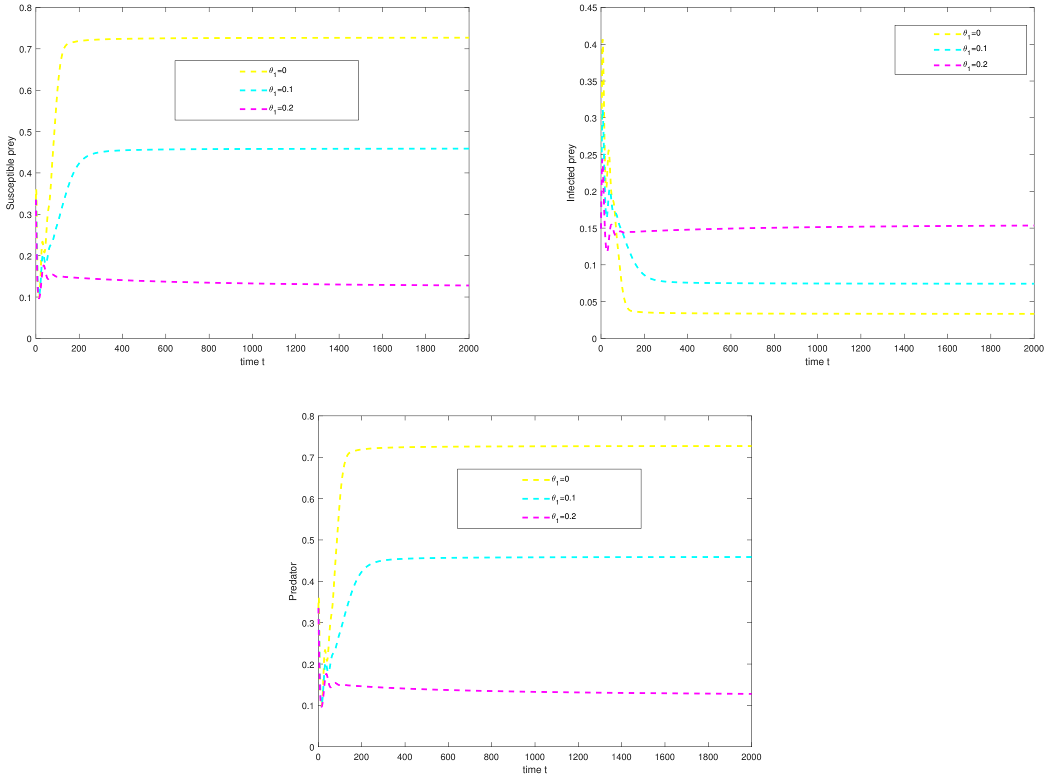


Figure 2. It is varying the harvesting rate of susceptible prey,infected prey and predator population with different values of $\theta_1 = 0, 0.1, 0.2$ for the fractional order derivative $\alpha = 0.92$.

6. Conclusions

In this study, we investigated a fractional-order derivative-based model of a three-species food web. Each equilibrium point's local stability in our proposed fractional-order system has also been examined. The suggested mathematical model's numerical simulation results show that the proposed system changes from unstable to stable as the order of the fractional derivative's value, α, goes from 0 to 1. It is obvious that for different values of α in the range $0 < \alpha < 1$, the unstable system with integer-order $\alpha = 1$ becomes a stable system. For the interior equilibrium point, when the derivative of $\alpha = 1$, the system becomes unstable, and if we change the order to fractional order, $\alpha = 0.92$, the system becomes stable. When the susceptible prey population harvesting rate increases, then the infected prey population harvesting rate decreases in the fractional order derivative. Since the susceptible prey population is inversely proportional to the infected prey population in the system. Consequently, the derivative of fractional-order alpha makes an important contribution to the proposed system's dynamical stability.

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