CONVERGENCE OF THE MANN ITERATION FOR HARDY–ROGERS CONTRACTION MAPPINGS IN CONVEX G_b-METRIC SPACES WITH AN APPLICATION TO FREDOLM-TYPE INTEGRAL EQUATIONS

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ABSTRACT

For a *Gb*-metric space (X,G,w); a contraction mapping T defined on X for which there exists only one x in X such that Tx = x; and a sequence in X generated by the Mann iteration scheme, under what conditions will this sequence converge to x, the unique fixed point of T in (X,G,w)? A theorem that answers this question for a Banach contraction mapping has been established, and inspired by it, a novel theorem is presented in this work that answers this question for a Hardy-Rogers contraction mapping which is a big extension of a Banach contraction mapping. Also, two novel corollaries are deduced from this theorem, one answering this question for a Kannan contraction mapping and the other doing so for a Chatterjea contraction mapping. Hence, this first theorem improves and extends the theorem which inspired it.

Next, attention is turned to the application of the theorem in proving the existence of a solution of a Fredolm-type Integral equation with two kernels and to this end a second theorem is presented. This type of integral equation arises in many problems in signal processing, theory of imaging, fluid mechanics, among other fields in the physical sciences and engineering. However, its solution is very difficult, if not impossible, to obtain by any of the existing analytical or numerical method for integral equation. Also, often when it describes a model, the guarantee that its solution exists is key to obtaining deductive insights from the model. Importantly, the second theorem presents a means of knowing if the solution of the equation exists which does not involve the very difficult task of solving the equation.

Theorem 1.0

If (X,G_b,W) be a complete convex G_b-metric space with constant s ≥ 0, and α_n a sequence in (0,1) that converges to $\alpha < \frac{1-s^3(a+b+c+d+e}{s^2}$, where a+b+c+d+e $< \frac{1}{s^3}$, then the sequence $\{x_n\}$ generated by the Mann iteration $W(x_n, Tx_n; \alpha_n) = x_{n+1}$ converges to a unique fixed point of the Hardy-Rogers contraction mapping T : X $\to$ X which statisfies the condition:

$$G(Tx,Ty,Ty) \leq aG(x,y,y) + bG(x,Tx,Tx) + cG(y,Ty,$$
$$+ dG(x,Ty,Ty) + eG(y,Tx,Tx)$$

for all x,y $\in$ X.

Corollary 1.0.1

If (X,G_b,W) be complete convex G_b-metric space with constant s ≥ 1, and $\{\alpha_n\}$ be a sequence in (0,1) that converges to $\alpha < \frac{1-s^2\beta(1+s)}{s(s+s\beta)}$, where $\beta \in \left[0, \frac{1}{s^2(s+1)}\right)$, then the sequence $\{x_n\}$ generated by the Mann iteration $W(x_n, Tx_n; \alpha_n) = x_{n+1}$; $x_0 \in X$, converges to a unique fixed point of the Chatterjea contraction mapping T : X $\to$ X satisfying the condition:

$$G(Tx,Ty,Ty) \leq \beta[G(x,Ty,Ty) + G(y,Tx,Tx)]$$

for all x,y $\in$ X

Corollary 1.0.2

If (X,G_b,W) be complete convex G_b-metric space with constant s ≥ 1, and $\{\alpha_n\}$ a sequence in (0,1) that converges to $\alpha < \frac{1-s\beta(s+1)}{s^2}$, where $\beta \in \left[0, \frac{1}{s(s+1)}\right)$, then the sequence $\{x_n\}$ generated by the Mann iteration $W(x_n, Tx_n; \alpha_n) = x_{n+1}$; $x_0 \in X$, converges to a unique fixed point of the Kannan contraction mapping T : X $\to$ X satisfying the condition:

$$G(Tx,Ty,Ty) \leq \beta[G(x,Tx,Tx) + G(y,Ty,Ty)]$$

for all x,y $\in$ X

Theorem 1.1(Application)

If the following conditions are satisfied :

(i) $\gamma \leq \frac{1}{2}$

(ii) $\int_a^b$ u(t,τ) ≤ 1

(iii) $|K_i(\tau, X(\tau)) - K_i(\tau, Y(\tau))| \leq \frac{\sqrt{17}}{17}|x-y|, i = 1,2$ and

(iv) $\int_a^b u(t,\tau)|K_1(\tau, y(\tau) + K_2(\tau, x(\tau))|d\tau \leq \sup\limits_{t\in[a,b]} \frac{|x(t)-Tx(t)|}{|x(t)-y(t)|}$

Then the integral equation

$$x(t) = f(t) + \gamma \int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau$$
$$\int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau \qquad \text{(A)}$$

where t $\in$ [a,b], and f: [a,b] $\to \mathbb{R}$, u: [a,b] x [a,b] $\to \mathbb{R}$ and $K_1, K_2 : [a,b] \to \mathbb{R}$ are all continuous functions on [a,b], has a unique solution in X = $(C[a,b], \mathbb{R})$

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