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Optimal Feedback Strategy For Dolichobrachristochrone Differential Game Problem Based On Dynamic Programming Approach

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INTRODUCTION & AIM

Differential game theory provides a powerful mathematical framework for analyzing the dynamic interactions between competing players by determining optimal strategies for each one. Originating from the pioneering work of Rufus Isaacs, this theory has evolved through numerous developments and criticisms, leading to more rigorous formulations based on dynamic programming and nonsmooth analysis.

In this work, we apply Miričă's dynamic programming approach to obtain a complete and rigorous solution of the classical Dolichobrachistochrone differential game, originally formulated by Isaacs. Our approach integrates theoretical results with numerical procedures to construct admissible and optimal feedback strategies. This framework not only refines the understanding of Isaacs's model but also offers practical tools applicable to robotics, aerospace, and economic systems where optimal decision-making in dynamic and competitive environments is essential.

DYNYNAMIC PROGRAMMING FORMULATION

Problem Given $w > 0$. Find:

$$\operatorname*{infsup}_{u(..)\,v(..)} C(y, u(..), v(..). \forall y \in Y_0,$$

with: $C(y, u(..) = g(x(T)) + \int_0^T f_0(x(t), u(t), v(t))dl,$

subject to:

$$x't) = f(x(t, u(t), v(t)),\ (0) = y, a.e.([0,T]),$$
$$u(t) \in U(a(t))\ a.e.([0,T), v(t) \in V(a(t))\ a.e.([0,T)).$$
$$x(t) \in Y_0, \forall/t \in [0,T), x(T) \in Y_1 T \textbf{ fixed.}$$

- **Defined by the foliowing data:**

$$f(a,u,v) = (\sqrt{x:}\rceil u_+ + {}^*w(v+1), \sqrt{2}(v-1)), \qquad f(o,u,v) = 1, £ = 0,$$
$$U = \{u \in \mathbb{R}2 : \|u\| = 1\}, \quad V = [-1,1]$$
$$Y_0 = (0.+\infty)^2, \quad Y_1 = \{0\} \times (0,+\infty)$$

GENERALIZED HAMILTONIAN AND CHARACTERISTIC FLOW

- If we denote by: $H_\pm(.,) = H(.,)\,|_{Z_\pm}, H_0(.,) = H(.,)\,|_{Z_0}$ follows:

$$\begin{aligned} H_+(x,p) &= -\sqrt{x_2}\,||p|| + w\,p_1 + 1, & if\ \ (x,p) \in Z_+ \\ H_-(x,p) &= -\sqrt{x_2}\,||p|| - w\,p_2 + 1, & if\ \ (x,p) \in Z_- \\ H_0(x,p) &= -\sqrt{2x_2}\,|p_1| + w\,p_1 + 1, & if\ \ (x,p) \in Z_0 \end{aligned}$$

- Set of terminal transversality points Z^* (see [3,4]) in our case is given

$$Z^* = \left\{ (0, s^2), (q_1, 0)) \,;\, q_1 = \frac{1}{s-}, s > w \right\}.$$

- The Hamiltonian system on the stratum Z_+ (which $h(x,p) = p_1 + p_2$

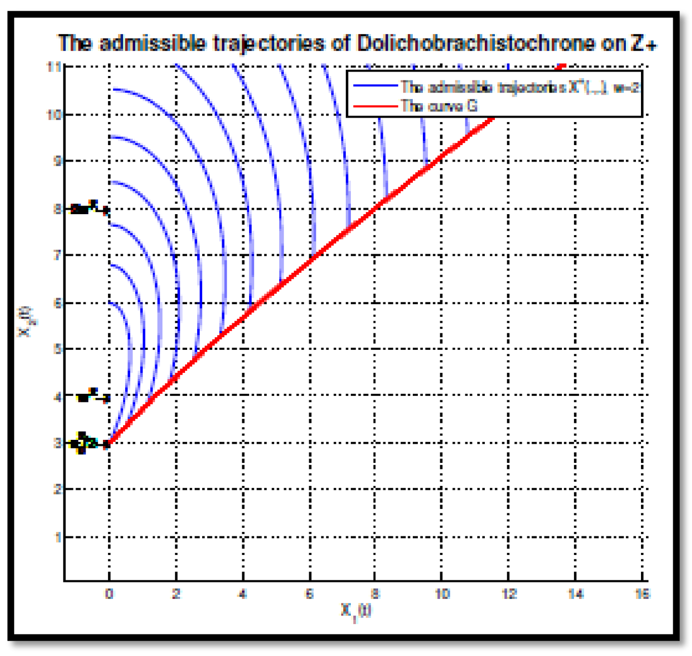
$$\begin{cases} x' = \left(-\sqrt{x_2}\,\frac{p_1}{||p||} + w,\ -\sqrt{x_2}\,\frac{p_2}{||p||}\right) \\ p' = \left(0,\ \frac{||p||}{2\sqrt{x_2}}\right). \end{cases}$$

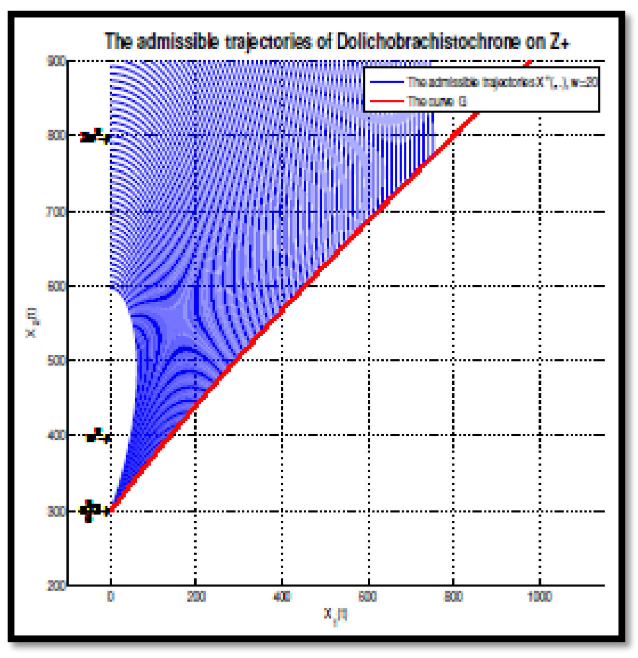
RESULTS & DISCUSSION

Theorem of optimality:
The following statements hold:
1. The function $W_0(.)$ of (91) in [2] is a solution of Isaacs' equation (59) in[2] on the corresponding domain $G = G+ \cup G- \cup G\ominus$. Moreover, it is the value function in the sense (55) of the corresponding admissible feedback strategies (92) in [2].
2. The corresponding admissible feedback stratigies U(.), $V(.)$ equation (92) in [] are optimal for the restriction on their domain G.

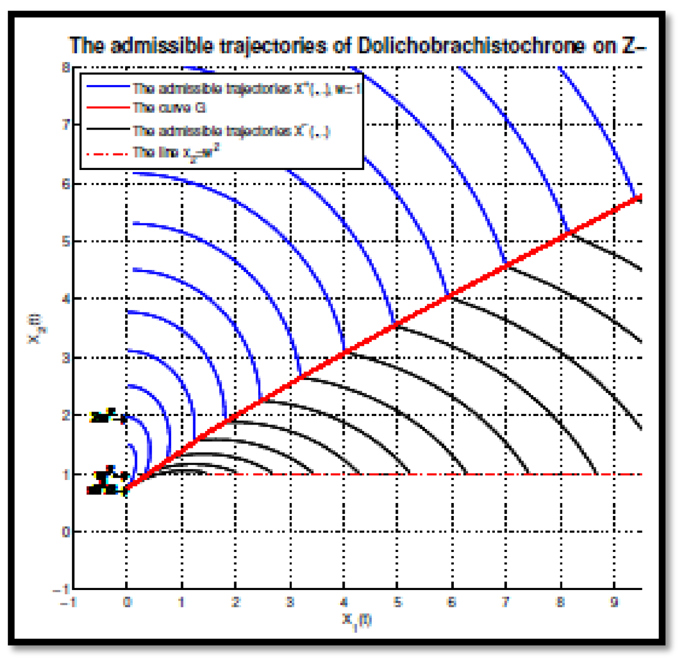
RESULTS & DISCUSSION

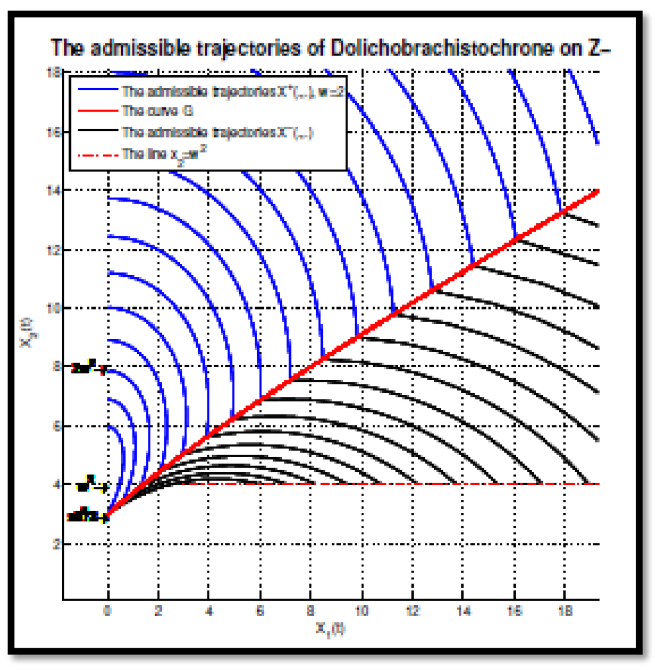
One may note here that, geometrically the trajectories X+ (., .) solution of the previous differential equation are the curves in Figure below and cover the domain Z+.



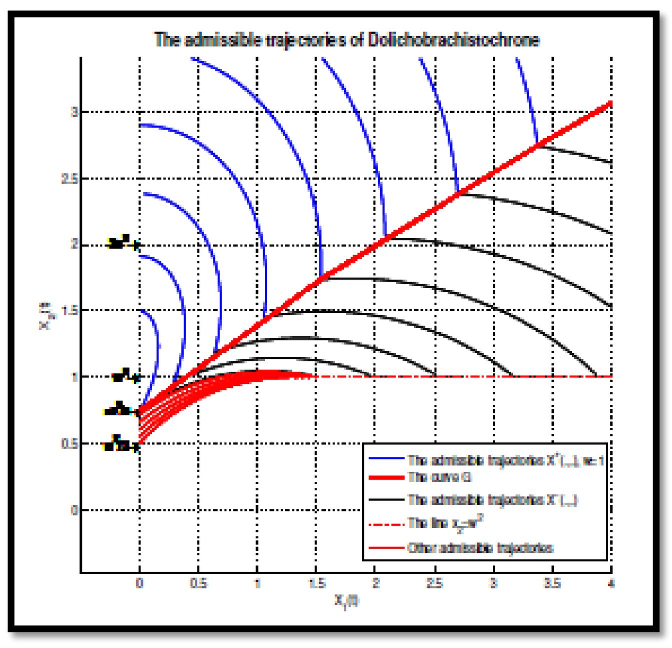


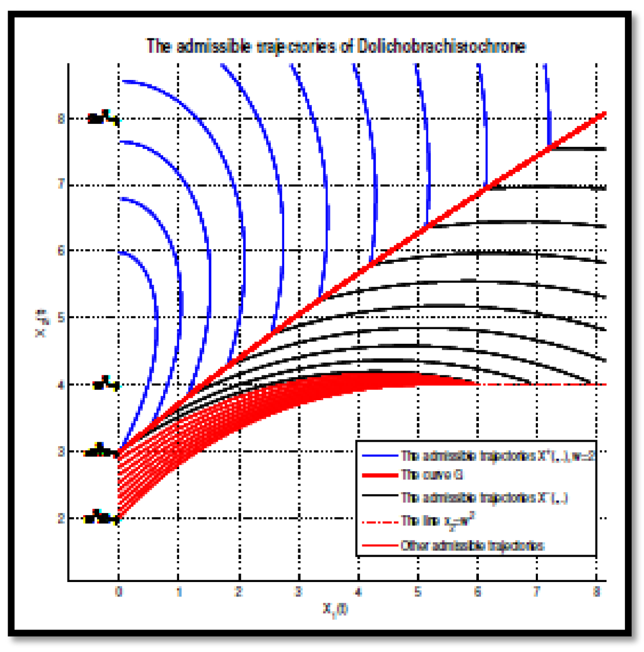
Belongs to the stratum Z+ and boundary of the open stratum Z−; examining the possibility of continuation for t < τ0 (s) of the trajectories X+(., s) , s > s0, we note that, this is possible only on the open stratum Z−.





A special role is played by other admissible trajectories of the system, which don't start from the terminal set Z− , but they start from the set in which H(x,p)=0. The Figure below shows the optimal Feedback Strategy For Dolichobrachistochrone problem.





CONCLUSION

A new and rigorous approach to the Dolichobrachistochrone differential game was developed using dynamic programming and non-smooth analysis. The results correct previous errors, identify the true barrier, and provide a clear framework for optimal feedback strategies.

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