Doubly Truncated Generalized Entropy

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5 November 2014

Overview

Abstract

Recently, the concept of generalized entropy has been proposed in the literature of information theory. In the present paper, we introduce and study the notion of generalized entropy in the interval (t_1, t_2) as uncertainty measure. It is shown that the suggested information measure uniquely determines the distribution function. Also, its properties has been studied. Some results have been obtained and some distributions such as uniform, exponential, Pareto, power series and finite range have been characterized by doubly truncated (interval) generalized entropy. Further, we describe a few orders based on this entropy and show its properties.

Introduction

In survival studies and life testing, information about the lifetime between two time points is available. In other words, event time of individuals which lies within a specific time interval are only observed. Thus, the analyzer cannot have access to the information about the subjects outside of this interval. For example, final products are often subject to selection checkup before being sent to the customer. The usual practice is that if a product's performance falls within certain tolerance limits, it is refereed compatible and sent to the customer. If it fails, a product is rejected and thus revoked. In this case, the actual distribution to the customer is called doubly (interval) truncated.

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dynamic uncertainty measure for two sided truncated random variables has been discussed by Sunoj et al. (2009), Misagh and Yari (2010) and Misagh and Yari (2011) as an extension of Shannon entropy. In this paper, an effort is made to develop some new characterizations to certain probability distributions and families of distributions using definition of doubly truncated generalized entropy which are suitable for modeling and analysis of lifetime data.

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isagh et al. (2010, 2011) consider the notion of interval entropy of random life time X in the interval (t_1, t_2) as an uncertainty measure contained in $(X|t_1 < X < t_2)$ as

$$IH(X, t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)} dx. \qquad (1)$$

Definition

- The first kind of generalized interval entropy of order β for a random lifetime X between time t_1 and t_2 is

$$IH_1^{\beta}(X, t_1, t_2) = \frac{1}{\beta - 1}\left(1 - \int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2) - F(t_1)}\right)^{\beta} dx\right) \qquad (2)$$

- The second kind of generalized interval entropy of order β for a random lifetime X between time t_1 and t_2 is

$$IH_2^{\beta}(X, t_1, t_2) = \frac{1}{1 - \beta}\log\left(\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2) - F(t_1)}\right)^{\beta} dx\right) \qquad (3)$$

where $f_X(x)$ is the probability density function of $X|t_1 < X < t_2$ and $(t_1, t_2) \in D = \left\{(u, v) \in R^{+^2}; F(u) \leq F(v)\right\}$.

Preliminaries

Example: Exponential distribution

Let X be a random variable with exponential distribution with survival function $\overline{F}(x) = e^{-\theta x}; x > 0$ then

$$IH_1^{\beta}(X, t_1, t_2) = \frac{1}{\beta - 1}\left(1 + \frac{1}{\theta\beta}\left[h_2^{\beta}(t_1, t_2) - h_1^{\beta}(t_1, t_2)\right]\right) \qquad (4)$$

and

$$IH_2^{\beta}(X, t_1, t_2) = \frac{1}{1 - \beta}\log\left(-\frac{1}{\theta\beta}\left[h_2^{\beta}(t_1, t_2) - h_1^{\beta}(t_1, t_2)\right]\right) \qquad (5)$$

where $h_j(t_1, t_2) = \frac{f\left(t_j\right)}{F(t_2) - F(t_1)}, j = 1, 2.$

Properties: Characterization of distribution function

Theorem

If X has an absolutely continuous distribution function $F(t)$ and if

- $IH_1^\beta(X, t_1, t_2)$ be increasing with respect to both coordinates t_1 and t_2, then $IH_1^\beta(X, t_1, t_2)$ uniquely determines $F(t)$.
- $IH_2^\beta(X, t_1, t_2)$ be increasing with respect to both coordinates t_1 and t_2, then $IH_2^\beta(X, t_1, t_2)$ uniquely determines $F(t)$.

Properties

Note

Since the generalized interval entropy determines the distribution function uniquely for each β, a natural question becomes apparent in this context is which β should be used in practice. The choice of β depends on the situation. For example, $IH_2^{\beta}(X, t_1, t_2)$ with $\beta = 2$ could be used as a measure of economic diversity in the context, of income analysis.

Theorem

The distribution of X is double truncated exponential if and only if $IH_1^{\beta}(X, t_1, t_2)(IH_2^{\beta}(X, t_1, t_2)) = c$, where c is a constant.

A few orders based on the generalized interval entropies

Proposition

Let X be an absolutely continuous random variable with density $f(x)$ and cumulative distribution function $F(x)$. Then

- increasing $h_1(t_1, t_2)$ in t_1 implies

$$IH_1^\beta(X, t_1, t_2) \leq \frac{1}{\beta - 1}\left(1 - h_1^\beta(t_1, t_2)\right) \tag{6}$$

  and

$$IH_2^\beta(X, t_1, t_2) \geq \frac{1}{1 - \beta}\log h_1^\beta(t_1, t_2) \tag{7}$$

- decreasing $h_2(t_1, t_2)$ in t_2 implies

$$IH_1^\beta(X, t_1, t_2) \leq \frac{1}{\beta - 1}\left(1 - h_2^\beta(t_1, t_2)\right) \tag{8}$$

and

$$IH_2^\beta(X, t_1, t_2) \geq \frac{1}{1 - \beta}\log h_2^\beta(t_1, t_2) \tag{9}$$

A few orders based on the generalized interval entropies

Definition

The random variable X is said to have

- decreasing first kind interval entropy or (DFIE) property if and only if for fixed t_2, $IH_1^{\beta}(X, t_1, t_2)$ is decreasing with respect to t_1.
- decreasing second kind interval entropy or (DSIE) property if and only if for fixed t_2, $IH_2^{\beta}(X, t_1, t_2)$ is decreasing with respect to t_1.

This implies that $IH_i^{\beta}(X, t_1, t_2)$; $i = 1, 2$, has DFIE(DSIE) if $\frac{\partial IH_i^{\beta}(X, t_1, t_2)}{\partial t_1} \leq 0$.

Theorem

If X is a nonnegative random variable then $IH_i^{\beta}(X, t_1, t_2)$; $i = 1, 2$ cannot be increasing function with respect to t_1 for any fixed t_2.

A few orders based on the generalized interval entropies

Theorem

Let X be a nonnegative random variable with probability density function $f(x)$ and cumulative function $F(x)$ then

$$\text{i) } IH_1^{\beta}(X, t_1, t_2) \leq \frac{1}{\beta - 1}\left(1 - \frac{1}{\beta}\left(\frac{1 + \frac{\partial \mu(t_1, t_2)}{\partial t_1}}{\mu(t_1, t_2)}\right)^{\beta - 1}\right) \tag{10}$$

and

$$\text{ii) } IH_2^{\beta}(X, t_1, t_2) \leq \frac{1}{\beta - 1} \log \frac{1}{\beta}\left(\frac{1 + \frac{\partial \mu(t_1, t_2)}{\partial t_1}}{\mu(t_1, t_2)}\right)^{\beta - 1} \tag{11}$$

where

$$\mu(t_1, t_2) = E(X - t | t_1 < X < t_2) = \frac{1}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} (z - t_1)\, dF(z) \tag{12}$$

is the doubly truncated mean residual life function.

Conclusion

In literature of information measures, generalized interval entropy is a famous concept which always give a nonnegative uncertainty measure. But in many survival studies for modeling statistical data, information about lifetime between two points is available. Considering, the concept of doubly truncated (interval) entropy has been introduced. In this paper, several results on the first and second kind of generalized interval entropies have been discussed. Also, it has been shown that generalized interval entropies determine the distribution of random variables uniquely. Some orders based on given uncertainty measures have been given.

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Acknowledgement