OPEN ACCESS

1st International Electronic Conference on Entropy and Its Applications
3 - 21 November 2014

www.sciforum.net/conference/ecea-1

Conference Proceedings Paper – Entropy

Application of Relative Entropy in Finding the Minimal Equivalent Martingale Measure(A note on MEMM)

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Received: 8 September 2014 / Accepted: 26 September 2014 / Published: 5 November 2014

Abstract: Minimal entropy martingale measure (MEMM) and geometric Levy process has been introduced as a pricing model for the incomplete financial market. This model has many good properties and is applicable to very wide classes of underlying asset price processes. MEMM is the nearest equivalent martingale measure to the original probability in the sense of Kullback-Leibler distance and is closely related to the large deviation theory .Those good properties has been explained. MEMM is also justified for option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion and Markov-modulated exponential Levy model.

Keywords: Relative entropy; Martingale measure; Minimal entropy martingale measure; Levy process; Markov-modulated

PACS Codes: ….

1. Introduction

The equivalent martingale measure method is one of the most powerful methods in the option pricing theory. If the market is complete, then the equivalent martingale measure is unique. On the other hand, in the incomplete market model there are many equivalent martingale measures. So we have to select

one equivalent martingale measure (EMM) as the suitable martingale measure in order to apply the martingale measure method.
Over the past three decades, academic researchers and market practitioners have developed and adopted different models and techniques for option valuation. The path-breaking work on option pricing was done by Black and Scholes [5] and Merton [31]. Föllmer and Sondermann [18], Föllmer and Schweizer [19] and Schweizer [39] determined a martingale pricing measure by minimizing a quadratic function of the losses from an imperfectly hedged position. Davis [10] considered a traditional economic argument for valuation, namely the marginal rate of substitution, and determined a pricing measure for option valuation by solving a utility maximization problem.
The well-known Black-Scholes model

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

is a very good model for the option pricing, even so this model has many week points, for example the gap between the historical volatility and the implied volatility, fat tail property and asymmetry property of the distribution of the log returns, etc. And so we need to introduce new models which may illustrate those properties. The geometric Lévy process model is one of them. This model is an incomplete market model, so there are many equivalent martingale measures. As the first candidate for the equivalent martingale measure the minimal martingale measure was introduced in [19]. After that several candidates have been offered, for example the Esscher martingale measure [20], the variance optimal martingale measure [40], the minimal entropy martingale measure [32] and etc.
In recent years, there is a considerable interest in the applications of regime switching models driven by a hidden Markov Chain process to various financial problems. For an overview of hidden Markov Chain processes and their financial applications, see Elliott et al. [11] and Elliott and Kopp [12]. Some works on the use of hidden Markov Chain models in finance include Elliott and van der Hoek [13] for asset allocation, Pliska [37] and Elliott et al. [14] for short rate models, Elliott and Hinz [16] for portfolio analysis and chart analysis, Guo [24] for option pricing under market incompleteness, Buffington and Elliott [6, 7] for pricing European and American options, Elliott et al. [17] for volatility estimation and the working paper by Elliott and Chan in 2004 for a dynamic portfolio selection problem. Much of the work in the literature focus on the use of the Esscher transform for option valuation under incomplete markets induced by Lévy-type processes. There is a relatively little amount of work on the use of the Esscher transform for option valuation under incomplete markets generated by other asset price dynamics, such as Markov regime switching processes. The market described by the Markov-modulated GBM model is incomplete in general, and, hence, the martingale measure is not unique. Instead of using the argument by Guo [24] that the market is completed by Arrow-Debreu securities, we can adopt the regime switching Esscher transform which is the modification of the random Esscher transform introduced by Siu et al. [41].It is assumed that the process of the parameters for the regime switching random Esscher transform is driven by the hidden Markov Chain model. By using the result in Miyahara [33], in [15] is justified the pricing result by the minimal entropy martingale measure, (MEMM). In this paper MEMM method is reviewed. Our paper organizes as follows: Section two presents the main idea of our paper. Geometric Lévy process and minimal entropy martingale measure pricing models is stated in section two. Section three considers Option pricing and MEMM under regime switching. The forth section is related to some other application of MEMM. Finally have been stated conclusion of the paper and proposes some topics for further investigation.

2. Geometric Lévy process and minimal entropy martingale measure pricing models

We assume that the value process of bond is given by

$$[B_t = \exp\{rt\}];$$

where r is a positive constant. A pricing model consists of the following two parts:
(A) The price process S_t of the underlying asset.
(B) The rule to compute the prices of options.

For the part (A) we adopt the geometric Lévy processes, so the part (A) is reduced to the selecting problem of a suitable class of the geometric Lévy processes. For the second part (B) we adopt the martingale measure method. then the price of an option X is given by $e^{-rt}E_Q[X]$. Our studies in this paper are carried on under such a framework.

2.1 *Geometric Lévy processes*

The price process S_t of a stock is assumed to be defined as what follows. We suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t; 0 \geq t \geq T\}$ are given, and that the price process $S_t = S_0 e^{Z_T}$ of a stock is defined on this probability space and given in the form

$$S_t = S_0 e^{Z_T} \qquad 0 \leq t \leq T$$

Where Z_t is a lévy process. Such a process S_t is named the geometric lévy process (GLP) and denoted the generating triplet of Z_t by $(\sigma^2, \nu(dx), b)$.

The price process S_t has the following another expression

$$S_t = S_0 \mathcal{E}(\tilde{Z})_t$$

where $\mathcal{E}(\tilde{Z})_t$ is the Doléans-Dade exponential (or stochastic exponential) of Z_t, and $\tilde{Z}_t$ is a Lévy process corresponding to the original Lévy process Z_t.

The generating triplet of $\tilde{Z}_t$, say $(\tilde{\sigma}^2, \tilde{\nu}(dx), \tilde{b})$, is

$$\tilde{\sigma}^2 = \sigma^2$$

$$\tilde{\nu}(dx) = (\nu \circ J^{-1})(dx), \quad J(x) = e^x - 1$$

$$\tilde{b} = b + \frac{1}{2}\sigma^2 + \int_{\{|x| \leq 1\}} (e^x - 1 - x)\nu(dx) + \int_{\{x < -1\}} (e^x - 1)\nu(dx) - \int_{\{log2 < x \leq 1\}} (e^x - 1)\nu(dx)$$

2.1.1 Simple return process and compound return process

As have be seen in the previous section, the GLP has two kinds of representation such that

$$S_t = S_0 e^{Z_T} = S_0 \mathcal{E}(\tilde{Z})_t$$

The processes Z_T and $\tilde{Z}_t$ are candidates for the risk process. In [35] is shown that

$$\Delta \tilde{Z}_k^{(n)} = \frac{\Delta S_k^{(n)}}{S_{k-1}^{(n)}} \quad , \quad \Delta Z_k^{(n)} = \log\left(1 + \frac{\Delta S_k^{(n)}}{S_{k-1}^{(n)}}\right)$$

Thus, $\Delta \tilde{Z}_k^{(n)}$ is the simple return process of $S_k^{(n)}$ and $\Delta Z_k^{(n)}$ is the increment of log-returns and it is called the compound return process of $S_k^{(n)}$.

2.2 *Equivalent martingale measures for GLP*

The candidates for the suitable equivalent martingale measure are as follows.

(1) Minimal Martingale Measure (MMM) (Föllmer-Schweizer(1991))
(2) Variance Optimal Martingale Measure (VOMM)(Schweizer(1995))
(3) Mean Correcting Martingale Measure (MCMM)
(4) Esscher Martingale Measure (ESMM) (Gerber-Shiu(1994), B-D-E-S(1996))
(5) Minimal Entropy Martingale Measure (MEMM) (Miyahara(1996), Frittelli(2000))
(6) Utility Based Martingale Measure (U-MM) [35]

2.2.1 Esscher transforms and Esscher Martingale Measure (ESMM)

Let $R_t, 0 \leq t \leq T$, be a stochastic process. Then the Esscher transformed measure of P by the risk process R_t and the index process h_s is the probability measure of $P^{(ESS)}_{R_{[0,T]},h}$ defined by

$$\frac{P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}}{dP}|_{\mathcal{F}} = \frac{e^{\int_0^T h_s dR_s}}{E[e^{\int_0^T h_s dR_s}]}$$

This measure transformation is called the Esscher transform by the risk process R_t and the index process h_s.

In the above definitions, if the index process is chosen so that the $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is a martingale measure of S_t, then $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is called the Esscher transformed martingale measure of S_t by the risk process R_t, and it is denoted by $P^{(ESS)}_{R_{[0,T]}}$or simply $P^{(ESS)}_R$.

2.2.2 Minimal entropy martingale measure (MEMM)

If an equivalent martingale measure P^* satisfies

$$H(P^*|P) \leq H(Q|P) \qquad \forall Q : \text{equivalent martingale measure;}$$

then P^* is called the minimal entropy martingale measure (MEMM) of S_t. Where $H(Q|P)$ is the relative entropy of Q with respect to P

$$H(Q|P) = \begin{cases} \int_\Omega \log\left[\frac{dQ}{dP}\right] dQ & , if\ Q \ll P \\ \infty, & otherwise, \end{cases}$$

Proposition: The simple return Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ of S_t is the minimal entropy martingale measure (MEMM) of S_t.

Remark: The uniqueness and existence theorems of ESMM and MEMM for geometric Lévy processes is proved in [34].

2.3 *Comparison of ESMM and MEMM*

a)Corresponding risk process: The risk process corresponding to the ESMM is the compound return process, and the risk process corresponding to the MEMM is the simple return process. The simple return process seems to be more essential in the relation to the original process rather than the compound return process.

b) Existence condition: For the existence of ESMM, $P^{(ESSM)}$, and MEMM, P^* , the following condition respectively is necessary.

$$\int_{\{|x|>1\}} |(e^x - 1)e^{h^* x}|\nu(dx) < \infty$$

$$\int_{\{|x|>1\}} |(e^x - 1)e^{\theta^*(e^x - 1)}|\nu(dx) < \infty$$

This means that the MEMM may be applied to the wider class of models than the ESMM. This difference does work in the stable process cases.

c) Corresponding utility function:The ESMM is corresponding to power utility function or logarithm utility function. On the other hand the MEMM is corresponding to the exponential utility function. the MEMM is very useful when one studies the valuation of contingent claims by (exponential) utility indifference valuation.

2.3.1 Properties special to MEMM

a) Minimal distance to the original probability:

The relative entropy is very popular in the field of information theory, and it is called Kullback-Leibler Information Number or Kullback-Leibler distance. Therefore we can state that the MEMM is the nearest equivalent martingale measure to the original probability P in the sense of Kullback-Leibler distance.

b) Large deviation property:

The large deviation theory is closely related to the minimum relative entropy analysis, and the Sanov's theorem or Sanov property is well-known. This theorem says that the MEMM is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures [34].

c)convergence question:

Several authors have proved in several settings and with various techniques that the minimal entropy martingale measure is the limit, as $p \searrow 1$, of the so-called p-optimal martingale measures obtained by minimizing the f-divergence associated to the function $f(y) = y^p$. This line of research was initiated in [21], [23], and later contributions include [28], [30], [42].

Apart from the above, there are a number of other areas where the minimal entropy martingale measure has come up [44]; these include

- option price comparisons [2], [3], [29], [26], [27], [36];
- generalizations or connections to other optimal ELMMs [1], [8], [9], [43];
- utility maximization with a random time horizon [4];
- good deal bounds [29];
- a calibration game [22].

3 Option pricing and Esscher transform under regime switching

In [15] is considered the option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM). That is, the market parameters, for instance, the market interest rate, the appreciation rate and the volatility of the underlying risky asset, depend on unobservable states of the economy which are modeled by a continuous-time Hidden Markov process. The market described by the Markov-modulated GBM model is incomplete in general and, hence, the martingale measure is not unique. They adopt a regime switching random Esscher transform to determine an equivalent martingale pricing measure. As in Miyahara [33], they can justify their pricing result by the minimal entropy martingale measure (MEMM).

3.1 *The model*

Suppose (Ω, F, P) is a complete probability space, where P is a real-world probability measure. Let T denote the time index set [0, T] of the model. Let $\{W_t\}_{t\in T}$ denote a standard Brownian Motion on (Ω, F, P). We assume that the states of the economy are modeled by a continuous-time hidden Markov Chain process $\{X_t\}_{t\in T}$ on (Ω, F, P) with a finite state space $X := (x_1, x_2, \dots, x_N)$. Without loss of generality, we can identify the state space of $\{X_t\}_{t\in T}$ to be a finite set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0) \in \mathcal{R}^N$. We suppose that $\{X_t\}_{t\in T}$ and $\{W_t\}_{t\in T}$ are independent.

Write $A(t)$ for the generator $[a_{ij}(t)]_{i,j=1,2,\dots,N}$. Then, from Elliott et al. [11], we have the following semi-martingale representation theorem for $\{X_t\}_{t\in T}$

$$X_t = X_0 + \int_0^t A(s)X_s ds + M_t$$

where $\{M_t\}_{t\in T}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the filtration generated by $\{X_t\}_{t\in T}$. We consider a financial model consisting of two risky underlying assets, namely a bank account and a stock, that are tradable continuously. The instantaneous market interest rate $\{r(t,X_t)\}_{t\in T}$ of the bank account is given by:

$$r_t := r(t,X_t) = \langle r, X_t\rangle ,$$

where $r := (r_1, r_2, \dots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \dots, N$ and $\langle .,. \rangle$ denotes the inner product in $\mathcal{R}^N$. In this case, the dynamics of the price process $\{B_t\}_{t\in T}$ for the bank account are described by:

$$dB_t = r_t B_t dt , B_0 = 1 .$$

We suppose that the stock appreciation rate $\{\mu_t\}_{t\in T}$ and the volatility $\{\sigma_t\}_{t\in T}$ of S also depend on $\{X_t\}_{t\in T}$ and are described by:

$$\mu_t := \mu(t,X_t) = \langle \mu, X_t\rangle , \sigma_t := \sigma(t,X_t) = \langle \sigma, X_t\rangle,$$

where $\mu := (\mu_1, \mu_2, \dots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \dots, N$. The dynamics of the stock price process $\{S_t\}_{t\in T}$ are then given by the following Markov-modulated Geometric Brownian Motion:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t , S_0 = s .$$

Let Z_t denote the logarithmic return $ln(S_t/S_0)$ from S over the interval $[0, t]$.
Then, the stock price dynamics can be written as:

$$S_t = S_u \, exp(Z_t - Z_u) ,$$

Where

$$Z_t = \int_0^t \left(\mu_s - \tfrac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s$$

Let $\theta_t := \theta(t,X_t)$ denote the regime switching Esscher parameter, which can be written as follows:

$$\theta(t,X_t) = \langle \theta, X_t\rangle$$

where $\theta := (\theta_1, \theta_2, \dots, \theta_N) \in \mathcal{R}^N$. Then, the regime switching Esscher transform $Q_\theta \sim P$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s\}_{s\in[0,t]}$ is given by:

$$\frac{dQ_\theta}{dP}|_{\mathcal{G}_t} = \frac{\exp\,(\int_0^t \theta_s dZ_s)}{E_P[\exp\,(\int_0^t \theta_s dZ_s)|\mathcal{F}_t^X]} , t \in T$$

Since $\int_0^t \theta_s dZ_s \,|\mathcal{F}_t^X \sim N(\int_0^t \theta_s(\mu_s - \frac{1}{2}\sigma_s^2)ds , \int_0^t \theta_s^2\sigma_s^2 ds)$.
The Radon-Nikodim derivative of the regime switching Esscher transform is given by:
The Radon-Nikodim derivative of the regime switching Esscher transform is given by:

$$\frac{dQ_\theta}{dP}|_{\mathcal{G}_t} = \exp\,(\int_0^t \theta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2\sigma_s^2 ds)$$

Here, we justify the choice of the equivalent martingale measure $Q_{\hat{\theta}}$ using the regime switching Esscher transform by minimizing the relative entropy with respect to P.
Proposition**:** Suppose there exists a β_t such that the following equation is satisfied:

$$\beta_t = \frac{r_t - \mu_t}{\sigma_t^2}$$

Let Q^* be a probability measure equivalent to the measure P on $\mathcal{G}_t$ defined by the following Radon-Nikodym derivative:

$$\frac{dQ^*}{dP}|_{\mathcal{G}_t} := \exp\left(\int_0^t \beta_s\sigma_s dW_s - \frac{1}{2}\int_0^t \beta_s^2\sigma_s^2 ds\right)$$

Then,

1. Q^* is well defined and uniquely determined by the above Radon-Nikodym derivative,
2. Q^* is the MEMM for the Markov-modulated GBM.

The proof of Proposition 3.1 is similar to the proof of Theorem 1 in Miyahara [33].

4. Some other application

In [38], is extended the result of Fujiwara (2009) to a general Markov-modulated Lévy model whose the main feature is the presence of a modulator factor which changes the characteristic of the dynamics of risky asset under different regimes. Their main contribution is to give an expression, when it exists, for the minimal entropy equivalent measure for this class of models. That work generalizes some previous works in the literature which have treated either the exponential Lévy case or the exponential additive case.

[9] deals with the existence and the explicit description of the minimal entropy–Hellinger local martingale density (called MH local martingale density). They show that this density is determined by point wise solution of equations in $\mathcal{R}^d$ depending only on the local characteristics of the discounted price process *S*. The uniform integrability as well as other integrability properties for the MH local martingale density is illustrated.

4. Conclusions

As we have seen, the MEMM has many good properties and seems to be superior to ESSMM from the theoretical point of view. And we can say that the [GLP & MEMM] model, which has been introduced in [21], is a strong candidate for the incomplete market model.
Developing method is found to price options when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM) based on a modification of the random Esscher transform by Siu et al. [41], namely the regime switching random Esscher transform. The choice of this martingale pricing measure is justified by the minimization of the relative entropy. Finally may explore the applications of these models to other types of exotic options or hybrid financial products, such as barrier options Asian options, game options and option-embedded insurance products, etc.

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