Scatter plots on the dynamical system of coupled nonlinear Klein-Gordon equations

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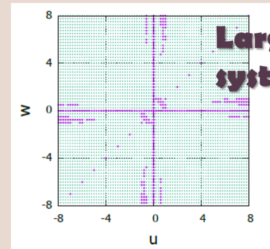
【Initial and Boundary Value Problem】 A coupled nonlinear Klein-Gordon equations:

$$\partial_t^2 u - \alpha_1 \partial_x^2 u - m_1 u + k_1 u^3 + k'_1 u^2 w = 0,$$
$$\partial_t^2 w - \alpha_2 \partial_x^2 w - m_2 w + k_2 w^3 + k'_2 u w^2 = 0,$$

Initial Cond.: $u(x, 0) = f(x),\ w(x, 0) = g(x),$
$\partial_t u\,(x, 0) = 0,\ \partial_t w\,(x, 0) = 0,$

Boundary Cond.: $u(0, t) = u(L, t),\ w(0, t) = w(L, t),$
$\partial_t u\,(0, t) = \partial_t u\,(L, t),\ \partial_t w\,(0, t) = \partial_t w\,(L, t)$



[This study] **Systematic Numerical Calculations** of **5012 cases**; each point stands for the initial values. **Green points** result in unbounded solutions, while **violet points** show the bounded solutions.

Preceding Works (Two Theoretical Milestones)

For a single equation with α_1, $\alpha_2 > 0$, the global existence of solution is ensured by **J. Ginibre and G. Velo (1985) [1]**,
For a small initial data, the conditional global existence of solution is proved by **H. Pecher (1984) [2]**:

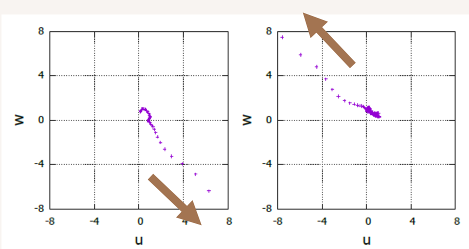
$k_1u^3 + k_1u^2w + k_2w^3 + k_2uw^2 > 0$ is true for any u and w.

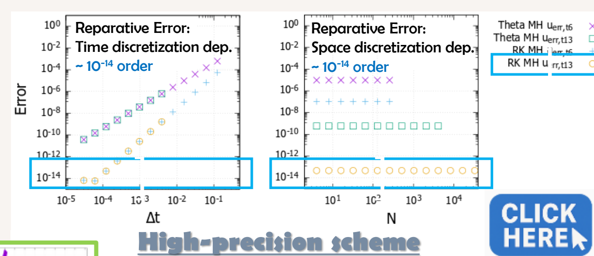
The present setting **potentially violates** the condition for **the global existence** of coupled equation due to the existence of nonlinear terms k'_1u^2w and k'_2uw^2. The inclusion of these terms could prevent the potential energy from being **positive definite** (or **bounded from below**), which is a crucial requirement for the global existence results by Pecher.

Result

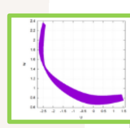
High-precision numerical scheme: implicit Runge-Kutta methods (time) with Spectral method (space)[3] is implemented.

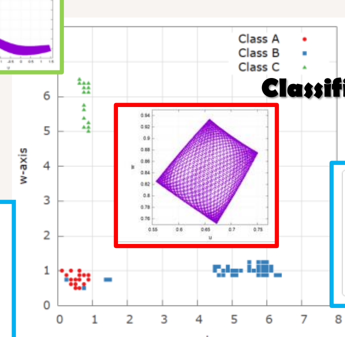
1. Using high-precision scheme, we found **many unbounded solutions (= blow-up solutions)** [4] and the condition:

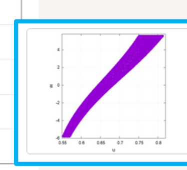


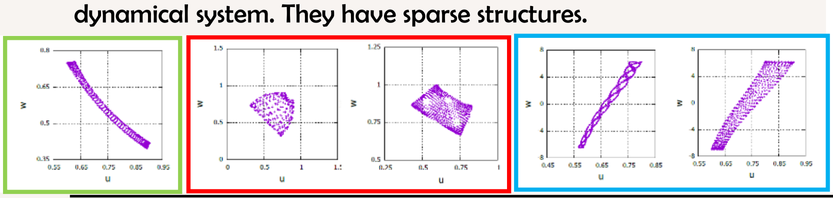


2. **[Gallery of Attractors]** More **artistic** than expected ! According to systematics, **bounded solutions** are **classified** into 3 types, and they are **visualized** in a finite dimensional dynamical system. They have sparse structures.









◆References:

[1] J. Ginibre and G. Velo, Mathematische Zeitschrift, Vol. 189 (1985), pp. 487-505.

[2] H. Pecher, Mathematische Zeitschrift, 186, 4 (1984) pp. 543--559.

[3] Y. Takei and Y. Iwata, Axioms 2022, 11(1), 28.

[4] Y. Takei and Y. Iwata, AIP Conf. Proc. 2872, 060026 (2023).