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Uniform Weak Convergence Rates for CTRW Approximations of Time-Fractional Diffusions with Unbounded Coefficients

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INTRODUCTION & AIM

Time-fractional diffusion equations admit a probabilistic representation through a diffusion process time-changed by the inverse of a stable subordinator. This makes continuous-time random walks (CTRWs) a natural probabilistic numerical method.

Existing weak convergence rate results for CTRW approximations are well understood in bounded-coefficient settings. We extend this theory to **unbounded but linearly growing coefficients**, a class that already includes geometric Brownian motion and fractional Black–Scholes models.

Main difficulty: Linearly growing coefficients break the natural framework tailored to the bounded coefficients [1] and force control of derivatives.

Aim. Prove explicit **uniform weak convergence rates** for CTRW approximations of time-fractional diffusions with linear-growth coefficients:

$$\partial_t^{\beta} u(t,x) = \frac{1}{\Gamma(1-\beta)}\, L\, u(t,x) - c u(t,x), \qquad u(0,x) = f(x),$$

$$L\, f(x) = b(x) f'(x) + \frac{1}{2}\sigma(x) f''(x).$$

METHOD

Our approximation combines a spatial Markov chain scheme for the diffusion with a heavy-tailed random walk approximation of the fractional clock.

Spatial model

$$d\, X_t = b(X_t)dt + \sigma(X_t) dW_t, \qquad X_0 = x.$$

The coefficients of this SDE are of linear growth with bounded derivatives up to the 4th order.

We approximate the spatial model by a discrete Markov process

$$Y_{n+1}^h = Y_n^h + b\big(Y_n^h\big)h + \sigma\big(Y_n^h\big)\sqrt{h}\,\xi, \qquad Y_0^h = x,$$

where ξ is a random variable with a symmetric distribution, $E\xi^2 = 1$ and $E\,\xi^4 < \infty$, and h is a step of the grid.

Temporal model

The time-change is given by an inverse β-stable subordinator $\hat{\sigma}_t$. The subordinator is approximated by a random walk with positive increments and heavy tails

$$\Phi_t^h = h^{1/\beta} \sum_{k=1}^{[\frac{t}{h}]} \tau_k\,, \qquad N_t^h = \sup\{s \in \mathbb{R}_+:\ \Phi_s^h \le t\},$$

with the density $p_{\tau(x)} \sim x^{-1-\text{\beta}}$ for large x. It is well-known that

$$\mathbb{E} e^{-c\hat{\sigma}_t}\, f\big(X_{\hat{\sigma}_t}\big) = u(t,x),$$

and, as such, we are motivated to consider an approximation $\mathbb{E}\, e^{N_t^h} f\big(Y^h_{N_t^h}\big)$ that we denote as $u_h(t,x)$. We establish uniform x in convergence rates.

We need to control not only the processes, but their derivatives as well. To this end, we apply the Kunita stochastic flow theory and consider the derivatives of X_t with respect to the initial data $DX_t(x)$. These derivatives satisfy the SDE

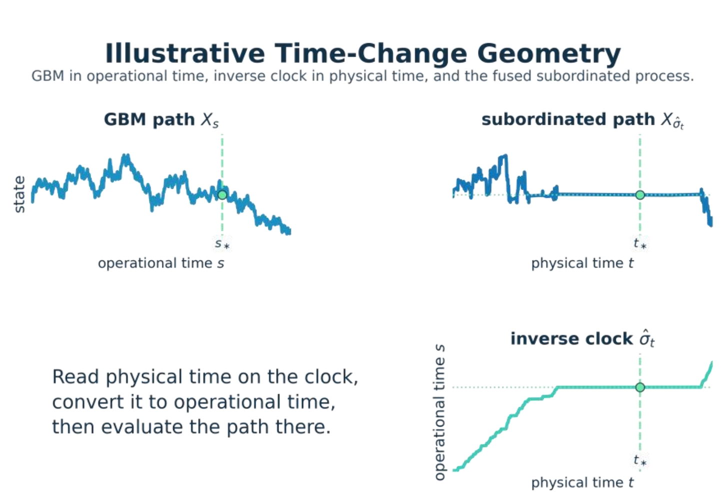
$$d\, DX_t = b'(X_t)\, DX_t dt + \sigma'(X_t)\, DX_t dW_t, \qquad DX_0 = 1.$$

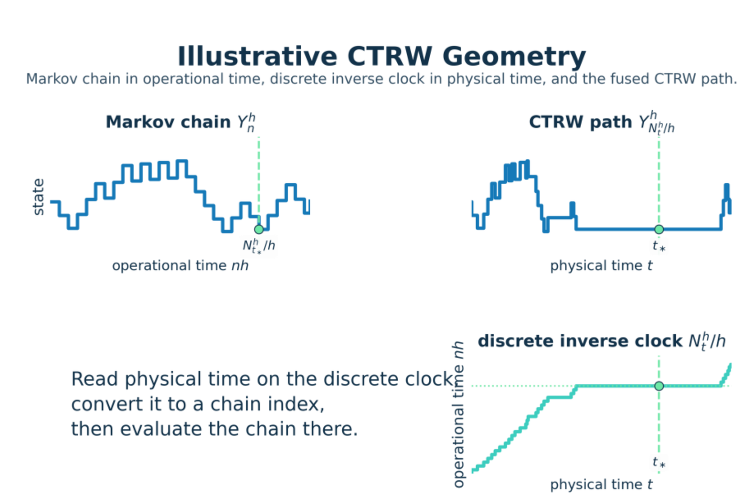
By the Faa di Bruno formula, the SDE for the higher order derivatives are written as well.

To obtain the convergence rates, we apply the semigroup theory. We employ the fact that the first differences of both continuous semigroup and its discretization approximate the generator. In particular, we obtain estimates for the derivatives,

$$\left|\frac{d^4}{dx^4}\ \mathbb{E}\, f(X_t(x))\right| \le Ce\text{^\{ct\}}\ ||f||_k$$

where $||f||_k$ is the norm in the space $C^k(\mathbb{R})$, and, as such, establish that the corresponding semigroup preserves smoothness with quantitative estimates.





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RESULTS & DISCUSSION

Theorem 1. Let f have bounded derivatives up to the fourth order. Then (μ can be found explicitly)

$$\sup_{0\le s\le t}\sup_x \frac{|\mathbb{E}_x f(X_t) - \mathbb{E}_x f(Y_t^h)|}{1+|x|^4} \le Ce^{\mu t} h\, ||\, f\, ||_4\,.$$

Theorem 2. Let f have bounded derivatives up to the fourth order. Then

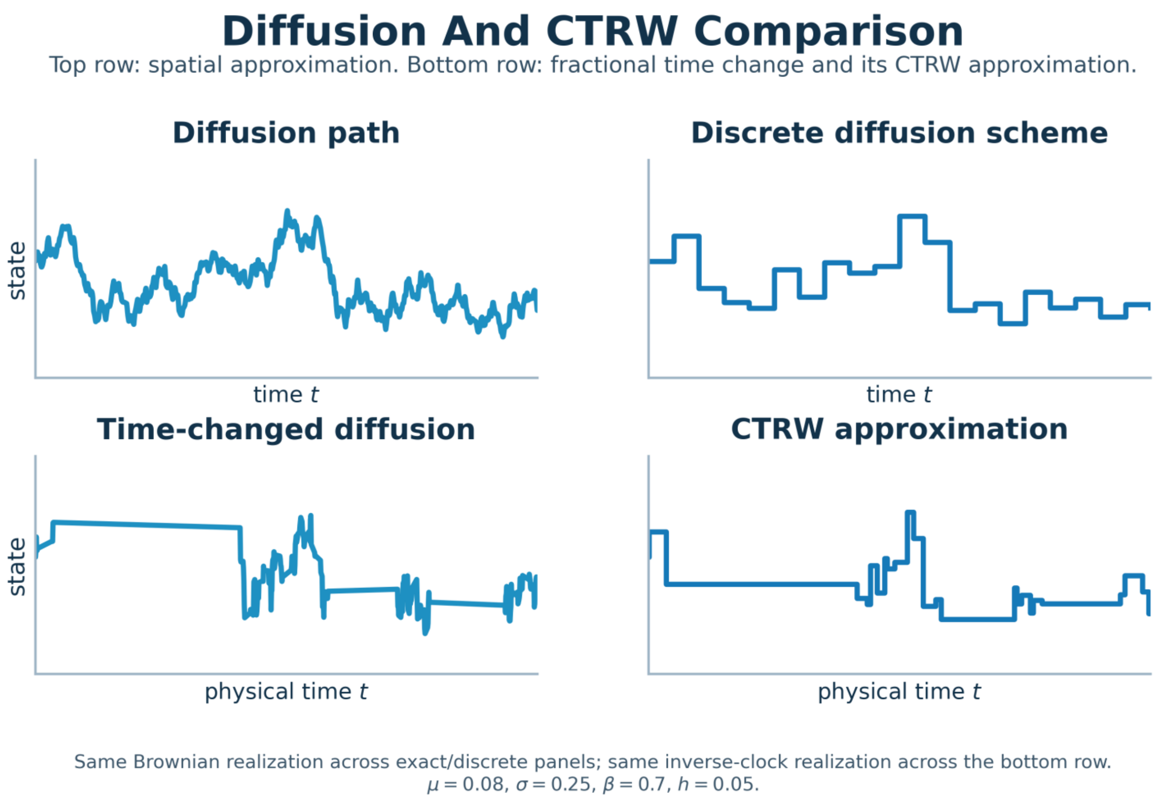
$$\sup_x \frac{|u_h(t,x) - u(t,x)|}{1+|x|^4} \le C\, log(1/h), \qquad \bar{\mu} < c < \mu,$$

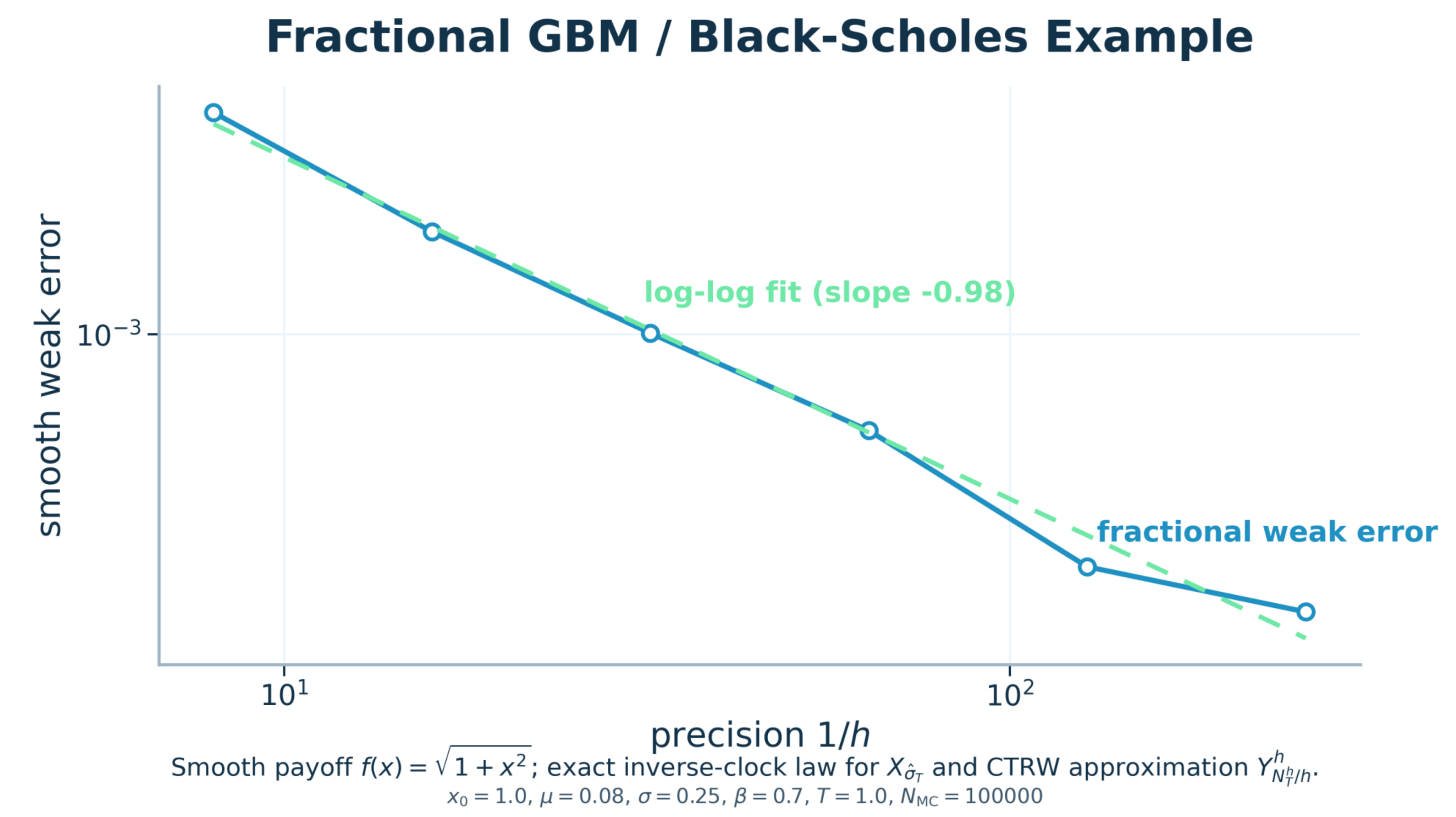
$$\sup_x \frac{|u_h(t,x) - u(t,x)|}{1+|x|^4} \le C\, h^{\chi(\beta)} + Ch^{1/(1+\beta)}, \qquad c = \mu,$$

$$\sup_x \frac{|u_h(t,x) - u(t,x)|}{1+|x|^4} \le C\, h^{\chi(\beta)}, \qquad c > \mu,$$

where μ and $\bar{\mu}$ can be found explicitly and $\chi(\beta)$ is taken from [1].

These results can be applied to compute convergence rates of numerical approximations to the fractional Black-Scholes equation. In this context, the change of time plays a role of the internal clock of the market. In practice, the probabilistic methods are used to approximate the paths as well, not only the solution to the Cauchy problem.





CONCLUSION

We extend CTRW weak convergence theory from bounded-coefficient models to diffusions with linearly growing coefficients. The key technical tool is a high-order stochastic-flow sensitivity framework, which yields:

- uniform spatial control of flow derivatives,
- weak one-step expansions strong enough for fractional CTRW approximation.

This gives a clean probabilistic route to time-fractional diffusion numerics in models beyond the bounded setting, including fractional Black–Scholes.

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