On a Boundary–Initial Value Problem for a Fractional Differential Equation with Sequential Caputo Derivatives

Jumaeva Shakhnoza*

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan

* **shahnozafarhodovna79@gmail.com**

Keywords: Sequential Caputo Derivatives, Bivariate Mittag-Leffler, Existence & Uniqueness, L1–FEM Graded Mesh

Abstract: *We investigate a fractional PDE with **sequential Caputo derivatives** $D_t^{\beta}(D_t^{\alpha}u)$, proving existence & uniqueness of the regular solution via the Fourier method. The analytic solution is expressed through the **bivariate Mittag-Leffler function** E_2. A robust L1–FEM scheme on a graded temporal mesh achieves convergence rates exceeding **1.14** in both L^2 and H^1 norms.*

Introduction & Problem Formulation

Fractional operators capture non-local memory and hereditary effects in anomalous diffusion, viscoelasticity, and epidemiology.

Sequential composition of two Caputo operators yields a fundamentally distinct structure (Miller & Ross, 1993):

$$\text{Non-commutativity}\, D_t^{\beta}(D_t^{\alpha}f) \neq D_t^{\alpha+\beta}f$$

Governing PDE. Seek $u : \Omega \times (0, T] \to \mathbb{R}$, $\Omega = (0, 1)$:

$$D_t^{\beta}(D_t^{\alpha}u(x,t)) + D_t^{\beta}u(x,t) - u_{xx}(x,t) = f(x,t)$$

with $0 < \alpha, \beta < 1$, subject to:

- **Initial:** $u(x, 0) = \varphi(x)$, $D_t^{\alpha}u(x, 0) = \psi(x)$
- **Boundary:** $u(0, t) = u(1, t) = 0$

Definition 1 (Regular Solution). *$u \in C(\bar{\Omega})$ is regular if $D_t^{\beta}(D_t^{\alpha}u)$, $u_{xx} \in C(\Omega)$ and $D_t^{\alpha}u \in C(\bar{\Omega})$, satisfying (1)–(4).*

Functional Framework & Bivariate Mittag-Leffler

Definition (Hölder Space $\overset{\circ}{C}{}^{a}[0,1]$). Let $a > \frac{1}{2}$. We denote by $\overset{\circ}{C}{}^{a}[0,1]$ the space of all functions $g \in C[0,1]$ satisfying the homogeneous boundary conditions $g(0) = g(1) = 0$ and the Hölder estimate

$$|g(x_1) - g(x_2)| \le C\,|x_1 - x_2|^{a}, \quad x_1, x_2 \in [0,1],$$

equipped with the norm

$$\|g\|_{\overset{\circ}{C}{}^{a}[0,1]} := \sup_{0 \le x_1 < x_2 \le 1} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|^{a}}.$$

The two-variable analogue is $f \in C(\bar{\Omega})$:

$$\overset{\circ}{C}{}^{a}{}_{x}(\bar{\Omega}) = \left\{ f : \omega_f(\delta; t) \le C(t)\,\delta^{a},\ f(0,t) = f(1,t) = 0 \right\},$$

where $\omega_f(\delta; t) = \sup_{|x_1 - x_2| \le \delta} |f(x_1, t) - f(x_2, t)|$ is the spatial modulus of continuity.

The analytic solution is expressed via the **bivariate Mittag-Leffler function** (Garg–Fernandez):

$$E_2(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m + \beta_1 n}(\gamma_2)_{\alpha_2 m}\, x^m y^n}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n)\,\Gamma(\delta_2 + \alpha_4 m)\,\Gamma(\delta_3 + \beta_3 n)}$$

Convergence: $\min\{\Delta_1, \Delta_2\} > 0$, where $\Delta_1 = \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2$, $\Delta_2 = \beta_2 + \beta_3 - \beta_1$.

Lemma 1 (Decay bound). Under $0 < \alpha < 2\beta < 2$, $\mu \le |\arg x| \le \pi$, $-K \le y \le 0$:

$$E_2(x,y) \le \frac{C}{1 + |x|}$$

Lemma 2 (Ashurov–Shakarova). For $a > \frac{1}{2}$, $g \in \overset{\circ}{C}{}^{a}[0,\pi]$:

$$\sum_{k=1}^{\infty} k^{\sigma}|g_k| \le C\|g\|_{\overset{\circ}{C}{}^{a}}, \quad \sigma \in \left[0,\, a - \tfrac{1}{2}\right)$$

Conclusions

- Well-posedness proved under $\alpha + \beta > 1$, $\alpha > \beta$ via the Fourier–E_2 method.
- Analytic formula: Explicit bivariate E_2 series with verified uniform convergence on $\bar{\Omega}$.
- Numerics: L1–FEM on graded mesh; rate > 1.1; error $\times 7.75$ vs. uniform at $N = 400$.
- Outlook: 2D domains, variable-order operators, inverse source problems.

Key References. [4] Miller & Ross (1993); [25] Kilbas, Srivastava & Trujillo (2006); [31] Fernandez, Kürt & Özarslan (2020); [32] Maes & Van Bockstal (2023).

MATLAB Code — Full simulation code released on *MATLAB Central File Exchange* (S. Jumaeva, 2026). **Conference:** `https://sciforum.net/event/iocff2026`

Main Theorem

Theorem 1 (Existence & Uniqueness). *Let $\alpha + \beta > 1$, $\alpha > \beta$, $a > \frac{1}{2}$, $\varphi, \psi \in \overset{\circ}{C}{}^{a}_{2}[0,1]$, $f \in \overset{\circ}{C}{}^{a}_{x}(\bar{\Omega})$. Then there exists a **unique regular solution**:*

$$u(x,t) = \sum_{k=1}^{\infty} U_k(t)\sin(k\pi x)$$

Proof strategy:

1. Fourier–Sine expansion reduces the PDE to a family of sequential fractional ODEs in $U_k(t)$.
2. Laplace + Prabhakar series yield explicit $U_k(t)$ via bivariate E_2 (Lemma 1.).
3. Weierstrass M-test + Lemma 2 confirm uniform convergence of u, u_{xx}, $D_t^{\alpha}u$, $D_t^{\beta}u$ on $\bar{\Omega}$.
4. Uniqueness follows from completeness of $\{\sin k\pi x\}$ in $L^2(0,1)$.

Lemma 3 The series $C\sum_{k=1}^{\infty} k^2\big(|\varphi_k| + |\psi_k| + \|f_k\|\big)$ converges uniformly on $\bar{\Omega}$ when $\varphi'', \psi'' \in \overset{\circ}{C}{}^{a}[0,1]$.

Numerical Scheme

Set $v(x,t) = D_t^{\alpha}u$ to decouple the nested operator:

Decoupled system

$$D_t^{\beta}v + D_t^{\beta}u - u_{xx} = f, \qquad v = D_t^{\alpha}u$$

Spatial: $\mathbb{P}_1$ conforming FEM on uniform mesh h; mass $M_{ij} = (\varphi_j, \varphi_i)$, stiffness $K_{ij} = (\varphi_j', \varphi_i')$.

Temporal: Graded mesh $t_n = T(n/N)^r$, $r = 2 - \min\{\alpha, \beta\}$; L1 weights:

$$a_{n,k}^{(\gamma)} = \frac{(t_n - t_{k-1})^{1-\gamma} - (t_n - t_k)^{1-\gamma}}{\Gamma(2-\gamma)\,\Delta t_k}$$

Implicit linear system at each step n:

$$A_n U^n = b^n, \quad A_n = K + c_n M, \quad c_n = a_{n,n}^{(\beta)}\big(1 + a_{n,n}^{(\alpha)}\big)$$

- **Unconditionally stable** fully-implicit scheme
- **Sparse** system at each time step
- **History splitting** keeps the right-hand side computable from previously computed values only

Manufactured-Solution Verification

Exact test: $g(x) = x^3(1-x)^3$, $u_{\text{ex}}(x,t) = g(x)\big(1 + t^{\alpha} + t^{\alpha+\beta}\big)$.

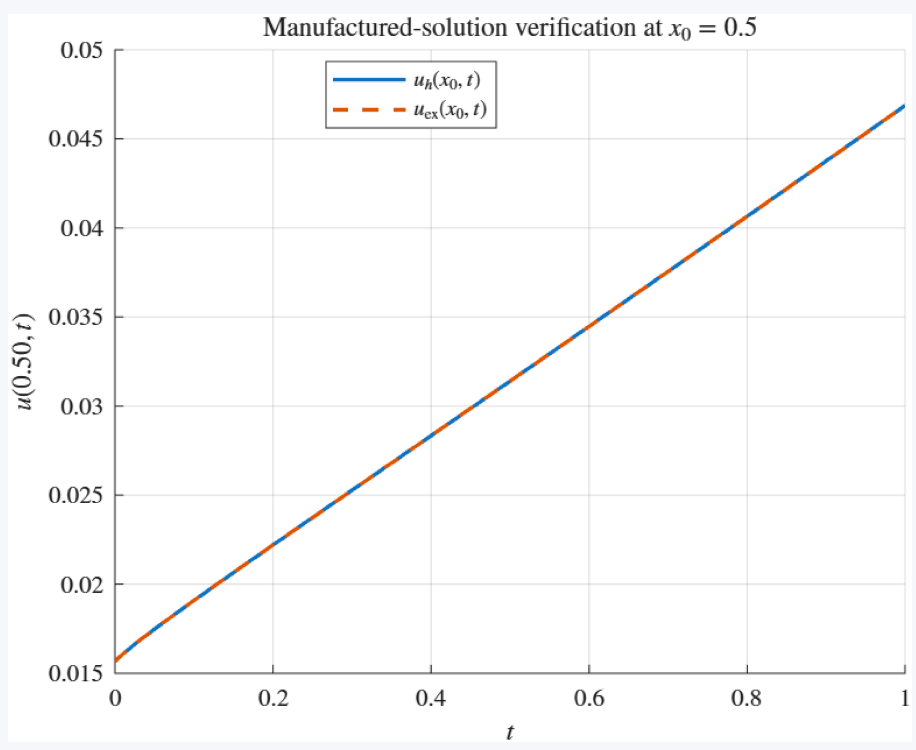

Figure 1: Exact u_{ex} vs numerical u_h trace at $x_0 = 0.5$. Curves overlap perfectly.

α	β	N	$\max_t \|e\|_{L^2}$	$\max_t \|e\|_{H^1}$
0.800	0.400	400	1.732×10^{-6}	6.344×10^{-6}

Table 1: Best result: $M = 200$, $r = 1.60$.

Fractional Parameter Dynamics

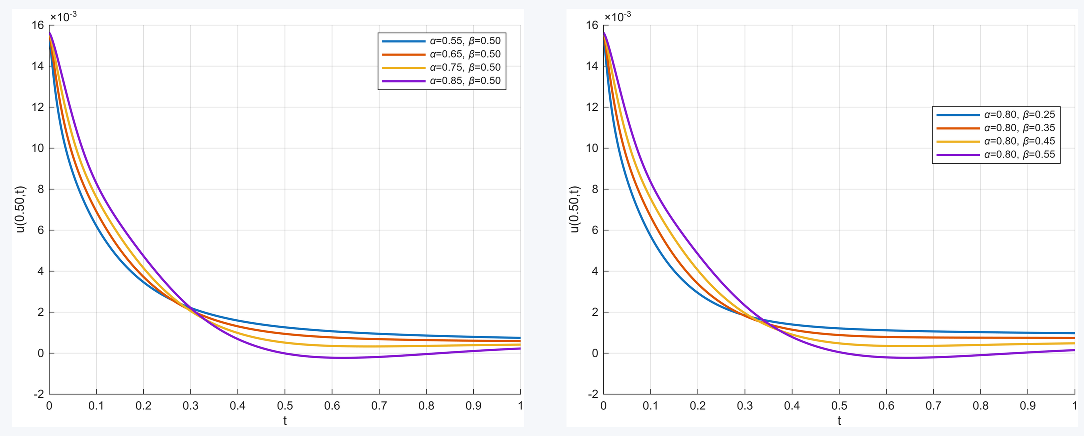

Figure 2: Pointwise traces $u_h(0.5, t)$. **Left:** effect of α (fixed $\beta = 0.50$). **Right:** effect of β (fixed $\alpha = 0.80$).

- Curves *cross* over time — parameters reshape the entire temporal profile, not merely its speed.
- Larger α or β induces a **sub-diffusive rebound** below zero: sequential-memory inertia.

Energy Decay

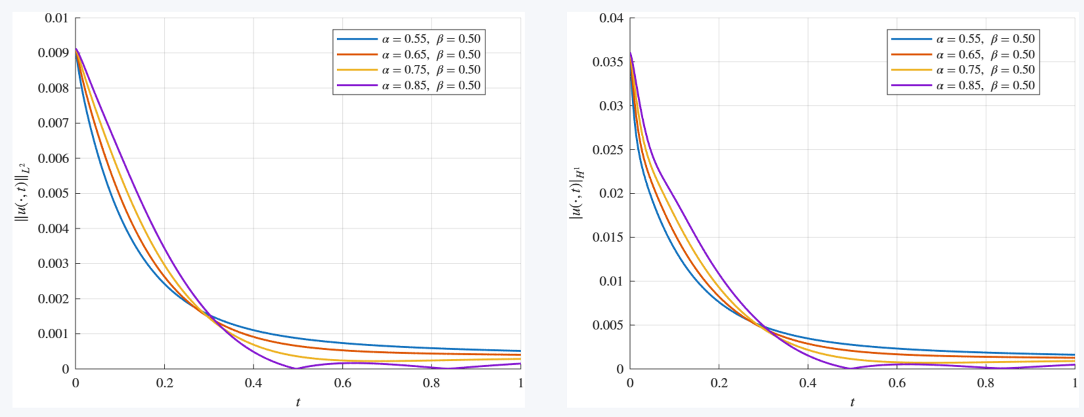

Figure 3: L^2-energy norm (left) and H^1-seminorm (right) for varying α, fixed $\beta = 0.50$.

Energy rebound. For large α, the L^2-norm drops to zero then slightly rebounds — a hallmark of *sequential-memory inertia*, analogous to a damped oscillator crossing equilibrium.

Parameter Heatmap

Diagnostic: $Q(\alpha,\beta) = u_h(0.5,\, t_0{=}0.5)$ over the admissible wedge $\{\alpha + \beta > 1,\ \alpha > \beta\}$.

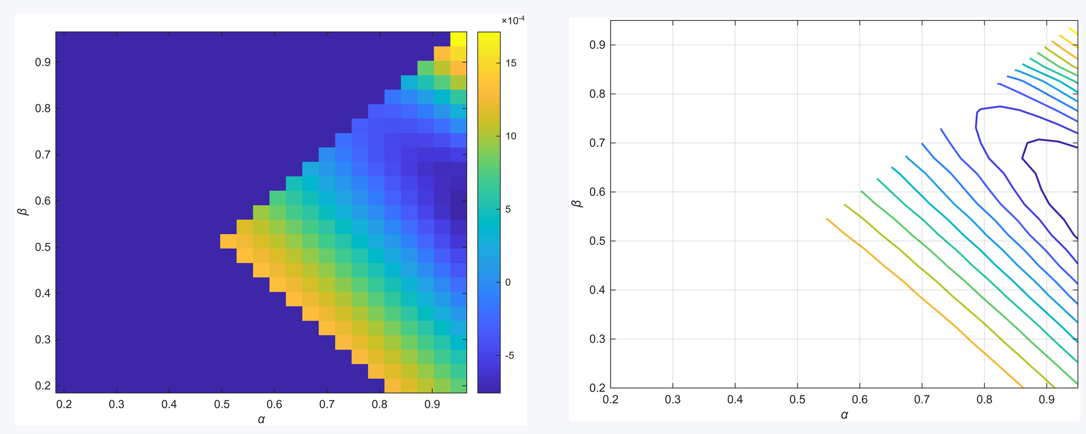

Figure 4: Heatmap of $Q(\alpha,\beta)$ (left) and iso-response contour curves (right).

Level curves reveal a **trade-off**: pairs (α,β) on the same contour produce identical outputs, enabling targeted parameter selection for physical models.

Convergence Analysis

Table 2: Uniform vs. graded mesh ($\alpha = 0.8$, $\beta = 0.4$, $M = 80$).

N	Mesh	$\max_t\|e\|_{L^2}$	$\max_t \|e\|_{H^1}$	Rate(L^2)	Rate(H^1)
100	Uniform ($r = 1$)	3.848×10^{-5}	1.331×10^{-4}	–	–
200	Uniform ($r = 1$)	2.297×10^{-5}	8.240×10^{-5}	0.744	0.692
400	Uniform ($r = 1$)	1.342×10^{-5}	4.972×10^{-5}	0.776	0.729
100	Graded (r=1.60)	8.290×10^{-6}	2.900×10^{-5}	–	–
200	Graded (r=1.60)	3.824×10^{-6}	1.374×10^{-5}	1.116	1.078
400	Graded (r=1.60)	**1.732e-6**	**6.344e-6**	**1.143**	**1.115**

Graded mesh restores rate $>$ 1.1 and cuts L^2-error by a factor of **7.75** at $N = 400$, compensating the initial weak singularity of $\partial_t^{\beta}(D_t^{\alpha}u)$ at $t = 0$.

LaTeX TikZposter