The quasi-integral Read-Bajraktarević functional, symmetries and Hardy-Orlicz spaces

The 1st International Online Conference on Fractal and Fractional
Recent Advances in Fractional-Order Differential and Integral Operators

Waldo Arriagada,
New Uzbekistan University Department of Mathematics

Define a convex, even, continuous map $\Phi : \mathbb{R} \to [0, +\infty)$:

$$\Phi(t) = \int_0^t \phi(s)\, ds. \qquad (1)$$

- $\Phi \in \Delta_2$ (global): for $r > 1$, $\exists\, \alpha(r) > 0$:

$$\Phi(rt) \leq \alpha(r)\, \Phi(t), \quad \text{for } t \geq 0. \qquad (2)$$

- Orlicz class $\mathcal{L}_\Phi(\Omega)$: measurable $u : \Omega \to \mathbb{R}$ st $\Phi(u) \in L^1(\Omega)$.

- In general, $\mathcal{L}_\Phi(\Omega)$ not a vector space. The linear hull $< \mathcal{L}_\Phi(\Omega) >= L_\Phi(\Omega)$ is the Orlicz space generated by Φ. Complete with the Luxemburg norm,

$$\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\}.$$

It follows that

$$L_\Phi(\Omega) = \left\{u : \Omega \to \mathbb{R} : \exists \lambda > 0, \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty\right\}. \qquad (3)$$

- The Orlicz class $\mathcal{H}^\Phi(\mathbb{D})$: set of (equivalence classes of) complex-valued functions f analytic on the (positive) unit disk $\mathbb{D} = \{z : |z| < 1\}$ for which

$$\sup_{0 \leq r < 1} \int_{|z|=1} \Phi(|f(rz)|) dz < \infty. \qquad (4)$$

- Fatou's lemma $\Rightarrow$ if $f \in \mathcal{H}^\Phi(\mathbb{D})$ then (4) coincides with

$$\int_{|z|=1} \Phi(|f(z)|) dz < \infty.$$

- In general, $\mathcal{H}^\Phi(\mathbb{D})$ is not a vector space. The Hardy-Orlicz space $H^\Phi(\mathbb{D})$ is the linear hull of the Hardy-Orlicz class (i.e., the set of analytic functions f such that $af \in \mathcal{H}^\Phi(\mathbb{D})$, for $a > 0$ which depends in general on f).

Then $H^\Phi(\mathbb{D})$ is Banach with respect to the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\tau > 0 : \int_{|z|=1} \Phi\left(\frac{1}{\tau}|f(z)|\right) dz \leq 1\right\},$$

cf. [Theorem 1.3.2, Leśniewicz 1971].

Hypothesis

There exist $p_\Phi, q_\Phi > 0$ such that

$$p_\Phi \leq \frac{t\,\phi(t)}{\Phi(t)} \leq q_\Phi < \infty, \qquad \text{for } t \neq 0. \qquad (5)$$

Consequences:

1) $\Phi \in \Delta_2$ globally [Adams & Fournier 2003].

2) Quick integration:

$$\min\{\nu^{p_\Phi}, \nu^{q_\Phi}\}\, \Phi(t) \leq \Phi(\nu t) \leq \max\{\nu^{p_\Phi}, \nu^{q_\Phi}\}\, \Phi(t) \qquad (6)$$

for all nonnegative real numbers ν and t.

Fix $n \in \mathbb{N}$ and define (real) families $\{\lambda_t : t \in [1, n+1]\}, \{R_t : t \in [1, n+1]\} \subset H^\Phi(\mathbb{D})$, depending continuously on t.

Let $\alpha : \mathbb{D} \to \mathbb{C}$ be a non-constant, one-to-one and real analytic conformal transformation on the (open) unit disk, with continuous extension to the boundary. We will assume that this map sends the unit circle into itself $\Longrightarrow \alpha(\mathrm{cl}(\mathbb{D})) = \mathrm{cl}(\mathbb{D})$.

Partition of $[-1, 1] = I_1 \cup \cdots \cup I_n$, with $I_i = [x_{i-1}, x_i)$, $1 \leq i \leq n-1$ and $I_n = [x_{n-1}, x_n = 1]$.

Partition $\mathrm{cl}(\mathbb{D})$ into real-symmetric vertical strips $X_i = \{z \in \mathrm{cl}(\mathbb{D}) : \mathrm{re}(z) \in I_i\}$. Then α induces real analytic bijections $\alpha_i : X_i \to \mathrm{cl}(\mathbb{D})$ sending $I_i \subset X_i$ into $[-1, 1]$.

$\Longrightarrow \{\alpha_i(X_i) : 1 \leq i \leq n\}$ is partition of the closed unit disk and

$$\alpha_i(X_i) \cap \alpha_{i'}(X_{i'}) = \emptyset \ \text{ for } i \neq i', \text{ and } \ \mathrm{cl}(\mathbb{D}) = \bigcup_{i=1}^n \alpha_i(X_i). \qquad (7)$$

(Quasi-integral) Read-Bajraktarević operator $\mathsf{T} : H^\Phi(\mathbb{D}) \to \mathbb{C}^{\mathbb{D}}$:

$$\mathsf{T}f(z) =$$

$$\sum_{i=1}^n \int_i^{i+1} \Big((\lambda_t \circ \alpha_i^{-1})(z) + (R_t \circ \alpha_i^{-1})(z)(f \circ \alpha_i^{-1})(z)\Big) \mathbb{1}_{\alpha_i(X_i)}(z) dt$$

(compare with [Jahn M., Massopust P., *An integral RB operator*, J. Fixed Point Theory Appl., 26-37 (2024).]

Definition

Local fractal function of the Hardy-Orlicz class $H^\Phi(\mathbb{D})$ is any fixed point $f^* \in H^\Phi(\mathbb{D})$ of the RBF.

Define constants

$$\Lambda = \max_{t \in [1,n+1],\, z \in \mathbb{S}^1} |\lambda_t(z)|, \quad M = \max_{z \in \mathbb{S}^1} |\alpha'(z)|, \quad L = \max_{t \in [1,n+1],\, z \in \mathbb{S}^1} |R_t(z)|,$$

where $\mathbb{S}^1$ is positive. Let $N = \max\{L^{p_\Phi}, L^{q_\Phi}\}$.

Theorem (Arriagada 2025)

The Read-Bajraktarević functional is well defined on $H^\Phi(\mathbb{D})$ and sends this space into itself. If $n^{q_\Phi} MN < 1$, then the operator is a contraction and its unique real fixed point is the limit of the iterates $f_k = \mathsf{T} f_{k-1}$, $k \in \mathbb{N}$, where $f_0 \in H^\Phi(\mathbb{D})$ is arbitrary.

For $1 \leq i \leq n$ and $u \in \mathrm{cl}(\mathbb{D})$ define $\nu_{i,u} : \mathrm{cl}(\mathbb{D}) \to \mathbb{C}$

$$\nu_{i,u}(z) = \int_i^{i+1} (\lambda_t(z) + R_t(z)\, u) dt. \qquad (8)$$

(MVT 1965: Uniformly Lipschitz continuous: for $z_1, z_2 \in \mathrm{cl}(\mathbb{D})$, $|\nu_{t,u}(z_2) - \nu_{t,u}(z_1)| \leq \tilde{M}|z_2 - z_1|$.) For $1 \leq i \leq n$ define

$$w_i : X_i \times \mathrm{cl}(\mathbb{D}) \to \mathrm{cl}(\mathbb{D}) \times \mathrm{cl}(\mathbb{D})$$
$$(z, u) \mapsto (\alpha_i(z), \nu_{i,u}(z)).$$

Set the metric $d_\theta : (\mathrm{cl}(\mathbb{D}) \times \mathrm{cl}(\mathbb{D})) \times (\mathrm{cl}(\mathbb{D}) \times \mathrm{cl}(\mathbb{D})) \to \mathbb{R}$

$$d_\theta((z_2, u_2), (z_1, u_1)) = |z_2 - z_1| + \theta|u_2 - u_1|,$$

where $\theta = (1 - M)/2\tilde{M}$.

Theorem (Arriagada 2025)

Let f^* be a fixed point of the RB functional. If $M < 1$, and $\Lambda + L < \frac{1}{n}$ then $\mathcal{F}_{\mathrm{loc}}^{\mathbb{C}} = \{(X_i \times \mathrm{cl}(\mathbb{D}), w_i) : 1 \leq i \leq n\}$ is a contractive local real CIFS for some metric, and $G(f^*)$ is a real-symmetric attractor of $\mathcal{F}_{\mathrm{loc}}^{\mathbb{C}}$.

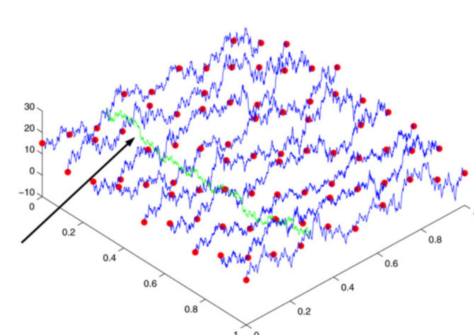

Figure: Taken from *P. Bouboulis, L. Dalla, Fractal Interpolation Surfaces derived from Fractal Interpolation Functions, J. Math. Anal. Appl. 336 (2007) 919–936*

References

[1] Arriagada W., *Local fractal functions on Orlicz-Sobolev spaces*, to appear, Studia Universitatis Babeș-Bolyai Mathematica, 2025.

[2] Arriagada W., *On local fractal functions of higher order*, Matematički Vesnik, 77, 3 (2025), 210-220.

[3] Bajraktarević M., *Sur une équation fonctionnelle*, Glasnik Mat. Fiz. I Astr. 12, 201–205 (1957).

[4] Jahn M., and Massopust P. *An integral RB operator*, J. Fixed Point Theory Appl., 26-37 (2024).

[5] Leśniewicz R. *On Hardy-Orlicz spaces I*, Annales Societatis Mathematicae Polonae Series I: Commentationes Mathematicae XV (1971).

[6] Read C.J., *A solution to the invariant subspace problem*, Bull. Lond. Math. Soc. **16**(4), 337–401 (1984).

Waldo Arriagada: New Uzbekistan University, Movarounnahr 1, Mirzo Ulugbek, Tashkent, Uzbekistan. Contact: w.arriagada@newuu.uz.