Certain sandwich-type results obtained for Atangana-Baleanu fractional integral applied to hypergeometric function

Alina Alb Lupaş[1], Daria Lupaş[2]
[1]Department of Mathematics and Computer Science
University of Oradea
1 Universitatii street, 410087 Oradea, Romania
[2] Faculty of Construction, Cadastre and Architecture,
University of Oradea, Romania
[1] alblupas@gmail.com, [2] daria.lupas@gmail.com

Abstract

Using the differential subordinations theory together with its dual, the differential superodinations theory, designed by Miller and Mocanu, we can reinterpret some inequalities concering real-valued functions regarding the complex-valued functions. As an overview, the subordination and superordination properties obtained for various operators, put together, generate sandwich results, as in this paper, where we define a new operator as the fractional integral developed by Atangana and Baleanu applied to confluent hypergeometric function and we describe several subordination results, which by duality give the corresponding superordination results and sandwich-type theorems.

Keywords: analytic function, univalent function, differential operator, fractional integral, differential subordination, best dominant, differential superordination, best subordinant.
2020 Mathematical Subject Classification: 30C45.

1 Introduction

The non-singular Mittag-Leffler kernel of the operator-defined Atangana-Baleanu fractional integral operator makes it of particular significance since it enables the operator to be employed in numerous applied mathematics fields for the development and investigation of mathematical models that incorporate it. By introducing the novel fractional derivative and the corresponding fractional integral with Mittag-Leffler kernel, the limitations of the conventional fractional-order derivatives including power-law kernel or exponential kernel were successfully eliminated. The power function or exponential function tend to be less appropriate for representing natural phenomena than the mittag-Leffler function. It occurs naturally in a number of physical problems as well as in science and engineering. Therefore, the Atangana-Baleanu fractional derivative and its corresponding Atangana-Baleanu fractional integral operator find extensive application in various fields, including probability theory, viscoelasticity, and groundwater fractal flow modeling.

The significance of the Mittag-Leffler function in fractional calculus is demonstrated in numerous works. In [1], various types of fractional evolution processes are described as being defined by fractional order equations, the solutions to which are associated with Mittag-Leffler-type functions. One can read a thorough overview of the Mittag-Leffler function's history and its applications in fractional analysis and fractional modeling, along with its generalizations, in [2]. More details about the Mittag-Leffler function's relevance within the framework of the Fractional Calculus is provided in [3], which starts with the analytical characteristics of the classical Mittag-Leffler function and moves on to the function's primary applications.

The disadvantage of using the Caputo fractional derivative to define the Caputo fractional integral operator is that the power kernel causes a singularity to occur at the interval's end point. First, a new non-singular fractional derivative with an exponential kernel was presented by Caputo and Fabrizio [4] in an attempt to solve this issue. By utilizing the generalized Mittag-Leffler function, Atangana and Baleanu enhanced the Caputo–Fabrizio fractional derivative with non-singular kernel and established the Atangana-Baleanu fractional derivative with non-local and non-singular kernel. A generalization of the Caputo-Fabrizio derivative is the Atangana-Baleanu fractional derivative. An additional development of the Caputo fractional derivative that utilizes the generalized hypergeometric type function is presented in [5].

The use of hypergeometric functions by L. de Branges in the famous Bieberbach conjecture proof [6] caused renewed interest in the study of hypergeometric functions in relationship with the theory of univalent functions. Confluent (Kummer) hypergeometric function has recently been examined from many angles. Its univalence conditions were outlined in [7], its applications on specific classes of univalent functions were demonstrated in [8], and an analytical investigation utilizing the fractional integral operator was carried out on Mittag-Leffler-confluent hypergeometric functions in [9].

The current research focuses on new uses of the Atangana-Baleanu fractional integral combined with confluent hypergeometric functions. An operator is defined using the Atagana-Baleanu fractional integral and the investigation using the differential subordination and superordination theories is the first approach on its applications. As a results of this investigation, sandwich-type results are stated. Section 2 of the paper presents the fundamental concepts and notations associated with those ideas, whereas Section 3 contains the original results.

2 Preliminaries

$\mathcal{H}(\Delta)$ denotes the set of analytic functions in $\Delta = \{\xi \in \mathbb{C} : |\xi| < 1\}$ the unit disc of $\mathbb{C}$. Two well known subsets of $\mathcal{H}(\Delta)$ are:

$$\mathcal{H}(a,n) = \left\{ g \in \mathcal{H}(\Delta) : g(\xi) = a + \sum_{j=n}^{\infty} a_j \xi^j,\ \xi \in \Delta \right\},$$

where $n \in \mathbb{N}$ and $a \in \mathbb{C}$,

and

$$\mathcal{A}_n = \{g \in \mathcal{H}(\Delta) : g(\xi) = \xi + \sum_{j=n+1}^{\infty} a_j \xi^j,\ \xi \in \Delta\},$$

marked with $\mathcal{A}$ for $n = 1$.

Atangana and Baleanu ([10]) introduced the left fractional integral with Mittag-Leffler kernel of order $\alpha \in [0,1]$ by

$$^{AB}_{\theta_1}I^{\alpha} f\left(\xi\right) = \frac{1-\alpha}{B(\alpha)} f(\xi) + \frac{\alpha}{\Gamma\left(\alpha\right) B(\alpha)} \int_{\theta_1}^{\xi} f\left(y\right) \left(\xi - y\right)^{\alpha-1} dy, \tag{2.1}$$

and Abdeljawad and Baleanu ([11]) introduced the right fractional integral with Mittag-Leffler kernel of order $\alpha \in [0,1]$ by

$$^{AB}I^{\alpha}_{\theta_2} f\left(\xi\right) = \frac{1-\alpha}{B(\alpha)} f(\xi) + \frac{\alpha}{\Gamma\left(\alpha\right) B(\alpha)} \int_{\xi}^{\theta_2} f\left(y\right) \left(y - \xi\right)^{\alpha-1} dy. \tag{2.2}$$

Kummer's differential equation

$$\xi f''\left(\xi\right) + \left(b - \xi\right) f'\left(\xi\right) - a f\left(\xi\right) = 0$$

is satisfies by confluent hypergeometric function ([12, p. 5]), which is analytic in $\mathbb{C}$ and is defined by relation

$$\phi\left(a, b; \xi\right) = 1 + \frac{a}{b}\frac{\xi}{1!} + \frac{a\left(a+1\right)}{b\left(b+1\right)}\frac{\xi^2}{2!} + ..., \; \xi \in \Delta, \tag{2.3}$$

where $a, b \in \mathbb{C}$ and $b \neq 0, -1, -2, ...$

Considering

$$\left(x\right)_j = \frac{\Gamma\left(x+j\right)}{\Gamma\left(x\right)} = x\left(x+1\right)\left(x+2\right)...\left(x+j-1\right) \text{ and } \left(x\right)_0 = 1,$$

then (2.3) can be expressed in the following form

$$\phi\left(a, b; \xi\right) = \sum_{j=0}^{\infty} \frac{\left(a\right)_j}{\left(b\right)_j}\frac{\xi^j}{j!} = \frac{\Gamma\left(b\right)}{\Gamma\left(a\right)} \sum_{j=0}^{\infty} \frac{\Gamma\left(a+j\right)}{\Gamma\left(b+j\right)}\frac{\xi^j}{j!}. \tag{2.4}$$

Next we remind the definitions of differential subordination and superordination. All functions from this article are supposed to be analytic

Definition 2.1 *([12]) The function h_1 is subordinate to the function h_2, or h_2 is superordinate to h_1, denoted $h_1 \prec h_2$, if there exists a Schwarz function f in U, with $f(0) = 0$ and $|f(\xi)| < 1$, $\forall\ \xi \in \Delta$, such that $h_1(\xi) = h_2(f(\xi))$, $\forall\ \xi \in \Delta$. When h_2 is univalent in U, the differential subordination is equivalent to $h_1(0) = h_2(0)$ and $h_1(\Delta) \subset h_2(\Delta)$.*

Definition 2.2 *([12]) For $f : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and g a univalent function in Δ, the function u in Δ which satisfies the differential subordination*

$$f(u(\xi), \xi u'(\xi), \xi^2 u''(\xi); \xi) \prec g(\xi), \quad \xi \in \Delta, \tag{2.5}$$

represents a solution of the differential subordination. The univalent function w represents a dominant of the solutions of (2.5) if $u \prec w$, $\forall\ u$ a solution of (2.5). A dominant $\widetilde{w}$ with the property $\widetilde{w} \prec w$ for all dominants w of (2.5) represents the best dominant of (2.5).

Definition 2.3 *([13]) For $f : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}$ and g a function in Δ, the univalent function u such that $f\left(u\left(\xi\right), \xi u'\left(\xi\right), \xi^2 u''\left(\xi\right); \xi\right)$ is univalent in Δ and satisfies the differential superordination*

$$g(\xi) \prec f(u(\xi), \xi u'(\xi), \xi^2 u''\left(\xi\right); \xi), \qquad \xi \in \Delta, \tag{2.6}$$

represents a solution of the differential superordination. A function w represents a subordinant of the solutions of (2.6) if $w \prec u$, $\forall$ u a solution of (2.6). A subordinant $\widetilde{w}$ with the property $w \prec \widetilde{w}$ for all subordinants w of (2.6) represents the best subordinant of (2.6).

Consider ([12]) $Q = \left\{f \in \mathcal{H}\left(\Delta\right) : f \text{ injective on } \overline{\Delta} \backslash E\left(f\right), \, f'\left(z\right) \neq 0, \, z \in \partial\Delta \backslash E\left(f\right)\right\}$, with $E\left(f\right) = \left\{z \in \partial\Delta : \lim\limits_{\xi \to z} f\left(\xi\right) = \infty\right\}$.

The results obtained in the paper are based on the following two lemmas.

Lemma 2.1 *([12]) Consider the univalent function w in Δ and the functions f and g in a domain $D \supset w\left(\Delta\right)$ with $g\left(z\right) \neq 0$ for $z \in w\left(\Delta\right)$. Define the functions $F\left(\xi\right) = \xi w'\left(\xi\right) g\left(w\left(\xi\right)\right)$ and $G\left(\xi\right) = f\left(w\left(\xi\right)\right) + F\left(\xi\right)$. In conditions:*

1) F is starlike univalent in Δ,

2) Re $\left(\frac{\xi G'(\xi)}{F(\xi)}\right) > 0$ for $\xi \in \Delta$,

3) the function u, having the properties $u\left(0\right) = w\left(0\right)$ and $u\left(\Delta\right) \subseteq D$, is a solution of the differential subordination

$$f\left(u\left(\xi\right)\right) + \xi u'\left(\xi\right) g\left(u\left(\xi\right)\right) \prec f\left(w\left(\xi\right)\right) + \xi w'\left(\xi\right) g\left(w\left(\xi\right)\right),$$

then the differential subordination holds

$$u\left(\xi\right) \prec w\left(\xi\right)$$

and w is best dominant.

Lemma 2.2 *([14]) Consider the convex univalent function w in Δ and the functions f and g in a domain $D \supset w\left(\Delta\right)$. In conditions:*

1) $F\left(\xi\right) = \xi w'\left(\xi\right) g\left(w\left(\xi\right)\right)$ is starlike univalent in Δ,

2) Re $\left(\frac{f'(w(\xi))}{g(w(\xi))}\right) > 0$ for $\xi \in \Delta$,

3) the function $f\left(u\left(\xi\right)\right) + \xi u'\left(\xi\right) g\left(u\left(\xi\right)\right)$ is univalent in Δ,

4) the function $u\left(\xi\right) \in \mathcal{H}\left[w\left(0\right), 1\right] \cap Q$, with $u\left(\Delta\right) \subseteq D$ which satisfies the differential superordination

$$f\left(w\left(\xi\right)\right) + \xi w'\left(\xi\right) g\left(w\left(\xi\right)\right) \prec f\left(u\left(\xi\right)\right) + \xi u'\left(\xi\right) g\left(u\left(\xi\right)\right),$$

then the differential superordination holds

$$w\left(\xi\right) \prec u\left(\xi\right)$$

and w is best subordinant.

3 Main Results

Applying the left fractional integral defined by Atangana and Baleanu for $\theta_1 = 0$ to hypergeometric function, we obtain a new operator studied in this paper.

Definition 3.1 *The left fractional integral defined by Atangana and Baleanu applied to confluent hypergeometric function* $\phi(a,b;\xi)$ *is defined by*

$$^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right) = \frac{1-\alpha}{B(\alpha)}\phi(a,b;\xi) + \frac{\alpha}{\Gamma(\alpha)B(\alpha)}\int_0^{\xi}\phi(a,b;y)(\xi-y)^{\alpha-1}dy, \tag{3.1}$$

where $\alpha \in [0,1]$, $a,b \in \mathbb{C}$, $b \neq 0,-1,-2,...$ *and* $\xi \in D\backslash\{0\}$.

Making several calculations, the left fractional integral defined by Atagana and Baleanu applied to confluent hypergeometric function can be written as

$$^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right) = \frac{1-\alpha}{B(\alpha)}\frac{\Gamma(b)}{\Gamma(a)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(b+j)\Gamma(j+1)}\xi^j + \frac{\alpha}{B(\alpha)}\frac{\Gamma(b)}{\Gamma(a)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(b+j)\Gamma(\alpha+j+1)}\xi^{j+\alpha},$$

or

$$^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right) = \frac{\Gamma(b)}{\Gamma(a)B(\alpha)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(b+j)}\left(\frac{1-\alpha}{\Gamma(j+1)} + \frac{\alpha\xi^{\alpha}}{\Gamma(\alpha+j+1)}\right)\xi^j. \tag{3.2}$$

We remark that $^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right) \in \mathcal{H}\left[\frac{1-\alpha}{B(\alpha)},\alpha\right]$.

The first subordination result obtained using the operator given by (3.1) is the next theorem:

Theorem 3.1 *Consider* $\left(\xi \cdot {}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)\right)^{\lambda} \in \mathcal{H}(\Delta)$ *and the univalent function* w *in* Δ *with the property* $w(\xi) \neq 0$, $\forall\ \xi \in \Delta$, *where* $a,b \in \mathbb{C}$, $b \neq 0,-1,-2,...$ *and* $\alpha \in [0,1]$, $\lambda > 0$. *Assuming that the function* $\frac{\xi w'(\xi)}{w(\xi)}$ *is starlike univalent in* Δ *and*

$$Re\left(1 + \frac{n}{q}w(\xi) + \frac{2p}{q}(w(\xi))^2 - \frac{\xi w'(\xi)}{w(\xi)} + \frac{\xi w''(\xi)}{w'(\xi)}\right) > 0, \tag{3.3}$$

for $m,n,p,q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$ *and*

$$H_{\alpha}^{a,b}(\lambda,m,n,p,q;\xi) := m + \lambda q + n\left(\xi \cdot {}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)\right)^{\lambda} + \tag{3.4}$$

$$p\left(\xi \cdot {}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)\right)^{2\lambda} + \lambda q\frac{\xi\left({}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)\right)'}{{}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)}.$$

If w *is a solution of the differential subordination*

$$H_{\alpha}^{a,b}(\lambda,m,n,p,q;\xi) \prec m + nw(\xi) + p(w(\xi))^2 + q\frac{\xi w'(\xi)}{w(\xi)}, \tag{3.5}$$

then w *is best dominant for the differential subordination*

$$\left(\xi \cdot {}^{AB}\ {}_0I^{\alpha}\left(\phi(a,b;\xi)\right)\right)^{\lambda} \prec w(\xi), \quad \xi \in \Delta. \tag{3.6}$$

Corollary 3.2 *Assuming that relation (3.3) takes place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...$ *and* $\alpha \in [0,1]$, $\lambda > 0$, *if the differential subordination*

$$H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi) \prec m + n\frac{M\xi + 1}{N\xi + 1} + p\left(\frac{M\xi + 1}{N\xi + 1}\right)^2 + q\frac{(M - N)\xi}{(M\xi + 1)(N\xi + 1)}$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $-1 \leq N < M \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\frac{M\xi+1}{N\xi+1}$ *is best dominant for the differential subordination*

$$\left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda} \prec \frac{M\xi + 1}{N\xi + 1}, \quad \xi \in \Delta.$$

Corollary 3.3 *Assuming that relation (3.3) takes place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...$ *and* $\alpha \in [0,1]$, $\lambda > 0$, *if the differential subordination*

$$H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi) \prec m + n\left(\frac{\xi + 1}{1 - \xi}\right)^r + p\left(\frac{\xi + 1}{1 - \xi}\right)^{2r} + q\frac{2r\xi}{1 - \xi^2}$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $0 < r \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\left(\frac{\xi+1}{1-\xi}\right)^r$ *is best dominant for the differential subordination*

$$\left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda} \prec \left(\frac{\xi + 1}{1 - \xi}\right)^r, \quad \xi \in \Delta.$$

Theorem 3.4 *Consider the univalent function* w *in* Δ *with the property* $w(\xi) \neq 0$, $\forall\ \xi \in \Delta$. *Assuming the function* $\frac{\xi w'(\xi)}{w(\xi)}$ *is starlike univalent in* Δ *and*

$$Re\left(\frac{2p}{q}(w(\xi))^2 + \frac{n}{q}w(\xi)\right) > 0, \text{ for } n, p, q \in \mathbb{C},\ q \neq 0. \tag{3.7}$$

If $\left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda} \in \mathcal{H}[0, (\alpha + 1)\lambda] \cap Q$, *the function* $H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi)$ *defined in (3.4) is univalent in* Δ, *where* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...$, $\alpha \in [0,1]$, $\lambda > 0$, *and the differential superordination*

$$m + nw(\xi) + p(w(\xi))^2 + q\frac{\xi w'(\xi)}{w(\xi)} \prec H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi) \tag{3.8}$$

is endowed for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, *then* w *is best subordinant for the differential superordination*

$$w(\xi) \prec \left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda}, \ \xi \in \Delta. \tag{3.9}$$

Corollary 3.5 *Assuming that relation (3.7) takes place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...$, $\alpha \in [0,1]$, $\lambda > 0$ *and* $\left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda} \in \mathcal{H}[0, (\alpha + 1)\lambda] \cap Q$, *if the differential superordination*

$$m + n\frac{M\xi + 1}{N\xi + 1} + p\left(\frac{M\xi + 1}{N\xi + 1}\right)^2 + q\frac{(M - N)\xi}{(M\xi + 1)(N\xi + 1)} \prec H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi)$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $-1 \leq N < M \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\frac{M\xi+1}{N\xi+1}$ *is best subordinant for the differential superordination*

$$\frac{M\xi + 1}{N\xi + 1} \prec \left(\xi \cdot^{AB} {}_0I^{\alpha}(\phi(a, b; \xi))\right)^{\lambda}, \quad \xi \in \Delta.$$

Corollary 3.6 *Assuming that relation (3.7) takes place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...,$ $\alpha \in [0,1]$, $\lambda > 0$, *and* $\left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda} \in \mathcal{H}\left[0,\left(\alpha+1\right)\lambda\right] \cap Q$, *if the differential superordination*

$$m + n\left(\frac{\xi+1}{1-\xi}\right)^{r} + p\left(\frac{\xi+1}{1-\xi}\right)^{2r} + q\frac{2r\xi}{1-\xi^{2}} \prec H_{\alpha}^{a,b}\left(\lambda, m, n, p, q; \xi\right)$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $0 < r \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\left(\frac{\xi+1}{1-\xi}\right)^{r}$ *is best subordinant for the differential superordination*

$$\left(\frac{\xi+1}{1-\xi}\right)^{r} \prec \left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda}, \ \ \xi \in \Delta.$$

Looking at Theorem 3.1 and Theorem 3.4 together, they generate a sandwich-type result.

Theorem 3.7 *Consider the univalent functions* w_1 *and* w_2 *in* Δ *with the properties* $w_1\left(\xi\right) \neq 0$, $w_2\left(\xi\right) \neq 0$, $\forall\ \xi \in \Delta$. *Assuming the functions* $\frac{\xi w_1'(\xi)}{w_1(\xi)}$ *and* $\frac{\xi w_2'(\xi)}{w_2(\xi)}$ *are starlike univalent in* Δ *and* w_1 *satisfies relation (3.3) and* w_2 *satisfies relation (3.7). If* $\left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda} \in \mathcal{H}\left[0,\left(\alpha+1\right)\lambda\right] \cap Q$, *the function* $H_{\alpha}^{a,b}\left(\lambda, m, n, p, q; \xi\right)$ *defined in (3.4) is univalent in* Δ, *where* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...,$ $\alpha \in [0,1]$, $\lambda > 0$, *and the sandwich-type result*

$$m + nw_1\left(\xi\right) + p\left(w_1\left(\xi\right)\right)^{2} + q\frac{\xi w_1'\left(\xi\right)}{w_1\left(\xi\right)} \prec H_{\alpha}^{a,b}\left(\lambda, m, n, p, q; \xi\right)$$

$$\prec m + nw_2\left(\xi\right) + p\left(w_2\left(\xi\right)\right)^{2} + q\frac{\xi w_2'\left(\xi\right)}{w_2\left(\xi\right)} \tag{3.10}$$

is endowed for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, *then* w_1 *and* w_2 *are respectively best subordinant and the best dominant for the following sandwich-type result*

$$w_1\left(\xi\right) \prec \left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda} \prec w_2\left(\xi\right), \ \ \xi \in \Delta. \tag{3.11}$$

For $w_1\left(\xi\right) = \frac{M_1\xi+1}{N_1\xi+1}$, $w_2\left(\xi\right) = \frac{M_2\xi+1}{N_2\xi+1}$, where $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, the following corollary is generated.

Corollary 3.8 *Assuming that relations (3.3) and (3.7) take place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...,$ $\alpha \in [0,1]$, $\lambda > 0$ *and* $\left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda} \in \mathcal{H}\left[0,\left(\alpha+1\right)\lambda\right] \cap Q$, *if the sandwich-type result*

$$m + n\frac{M_1\xi+1}{N_1\xi+1} + p\left(\frac{M_1\xi+1}{N_1\xi+1}\right)^{2} + q\frac{\left(M_1-N_1\right)\xi}{\left(M_1\xi+1\right)\left(N_1\xi+1\right)} \prec H_{\alpha}^{a,b}\left(\lambda, m, n, p, q; \xi\right)$$

$$\prec m + n\frac{M_2\xi+1}{N_2\xi+1} + p\left(\frac{M_2\xi+1}{N_2\xi+1}\right)^{2} + q\frac{\left(M_2-N_2\right)\xi}{\left(M_2\xi+1\right)\left(N_2\xi+1\right)}$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\frac{M_1\xi+1}{N_1\xi+1}$ *and* $\frac{M_2\xi+1}{N_2\xi+1}$ *are respectively best subordinant and best dominant for the following sandwich-type result*

$$\frac{M_1\xi+1}{N_1\xi+1} \prec \left(\xi \cdot^{AB}\ {}_0I^{\alpha}\left(\phi\left(a,b;\xi\right)\right)\right)^{\lambda} \prec \frac{M_2\xi+1}{N_2\xi+1}, \quad \xi \in \Delta.$$

For $w_1(\xi) = \left(\frac{\xi+1}{1-\xi}\right)^{r_1}$, $w_2(\xi) = \left(\frac{\xi+1}{1-\xi}\right)^{r_2}$, where $0 < r_1 < r_2 \leq 1$, the following corollary is generated.

Corollary 3.9 *Assuming that relations (3.3) and (3.7) take place for* $a, b \in \mathbb{C}$, $b \neq 0, -1, -2, ...$, $\alpha \in [0,1]$, $\lambda > 0$, *and* $\left(\xi \cdot^{AB}\ {}_0I^{\alpha}(\phi(a,b;\xi))\right)^{\lambda} \in \mathcal{H}[0, (\alpha+1)\lambda] \cap Q$, *if the sandwich-type result*

$$m + n\left(\frac{\xi+1}{1-\xi}\right)^{r_1} + p\left(\frac{\xi+1}{1-\xi}\right)^{2r_1} + q\frac{2r_1\xi}{1-\xi^2} \prec H_{\alpha}^{a,b}(\lambda, m, n, p, q; \xi)$$

$$\prec m + n\left(\frac{\xi+1}{1-\xi}\right)^{r_2} + p\left(\frac{\xi+1}{1-\xi}\right)^{2r_2} + q\frac{2r_2\xi}{1-\xi^2}$$

is fulfilled for $m, n, p, q \in \mathbb{C}$, $q \neq 0$, $\xi \in \Delta$, $0 < r_1 < r_2 \leq 1$, *and the function* $H_{\alpha}^{a,b}$ *is defined by relation (3.4), then* $\left(\frac{\xi+1}{1-\xi}\right)^{r_1}$ *and* $\left(\frac{\xi+1}{1-\xi}\right)^{r_2}$ *are respectively best subordinant and best dominant for the following sandwich-type result*

$$\left(\frac{\xi+1}{1-\xi}\right)^{r_1} \prec \left(\xi \cdot^{AB}\ {}_0I^{\alpha}(\phi(a,b;\xi))\right)^{\lambda} \prec \left(\frac{\xi+1}{1-\xi}\right)^{r_2}, \quad \xi \in \Delta.$$

4 Conclusion

This investigation considers the confluent hypergeometric function and introduces the Atagana-Baleanu fractional integral of the confluent hypergeometric function. The results that were obtained by combining fractional integral with specific hypergeometric functions acted as motivation for this development. By applying it, a new operator is introduced and the theory of differential subordination is employed in order to derive interesting subordinations and to provide their best dominants. For some well-known functions that are applied as the best dominants of the subordinations under investigation, some nice corollaries are given. Various differential superordinations are also established using the operator introduced by employing the Atagana-Baleanu fractional integral of the confluent hypergeometric function, and their best subordinants are also given. A sandwich-type result links the subordination and superordination conclusions of Theorems 3.1 and 3.4. For certain functions employed as best dominant and best subordinant in the sandwich-type result, interesting corollaries follow. The investigation conducted in this paper may encourage the use of different hypergeometric functions combined with the Atagana-Baleanu fractional integral. Additionally, conditions for univalence of the operator provided here could be studied given the best subordinants of the differential subordinations obtained in this study. Additional research using different approaches may be carried out regarding the operator presented in (3.1).

References

[1] Mainardi, F.; Gorenflo, R. On Mittag-Leffler-type functions in fractional evolution processes. *Journal of Computational and Applied Mathematics*, **2000**, *118 (1–2)*, 283–299. https://doi.org/10.1016/S0377-0427(00)00294-6

[2] Rogosin, S. The Role of the Mittag-Leffler Function in Fractional Modeling. *Mathematics*, **2015**, *3(2)*, 368–381; https://doi.org/10.3390/math3020368

[3] Mainardi, F. Why the Mittag-Leffler Function Can Be Considered the Queen Function of the Fractional Calculus?. *Entropy*, **2020**, *22*, 1359. https://doi.org/10.3390/e22121359

[4] Caputo, M.; Fabrizio, M. A New definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.*, **2015**, *1*, 73–85.

[5] Agarwal, P.; Jain, S.; Mansour, T. Further extended Caputo fractional derivative operator and its applications. *Russ. J. Math. Phys.* **2017**, *24*, 415–425. https://doi.org/10.1134/S106192081704001X

[6] De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152.

[7] Oros, G.I. New Conditions for Univalence of Confluent Hypergeometric Function. *Symmetry* **2021**, *13(1)*, 82.

[8] Porwal, S.; Kumar, S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions. *Afr. Mat.* **2017**, *28*, 1–8.

[9] Ghanim, F.; Al-Janaby, H. F. An analytical study on Mittag-Leffler–confluent hypergeometric functions with fractional integral operator. *Math. Methods Appl. Sci.*, **2020**, https://doi.org/10.1002/mma.6966.

[10] Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Science*, **2016**, *20(2)*, 763–769.

[11] Abdeljawad, T.; Baleanu, D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *The Journal of Nonlinear Sciences and Applications*, **2017**, *10(3)*, 1098–1107.

[12] Miller, S.S.; Mocanu, P.T. Differential Subordinations. Theory and Applications. Marcel Dekker Inc., New York, Basel, 2000.

[13] Miller, S.S.; Mocanu, P.T. Subordinants of differential superordinations. *Complex Var.* **2003**, *48*, 815–826.

[14] Bulboacă, T. Classes of first order differential superordinations. *Demonstratio Math.* **2002**, *35(2)*, 287–292.