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On the Analytical Treatment of Fractional Elliptic Equations: Applications to Steady-State Heat Conduction

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INTRODUCTION

This work models steady-state heat conduction in a circular plate, as illustrated in Figure 1. The Riemann–Liouville and Caputo definitions are adopted to obtain the solutions when fractional order is taken into account. The problem is governed by the generalized partial differential equation (PDE) as follows:

$$\nabla^{2\alpha} u(\rho,\varphi) = D_\rho^{2\alpha} u(\rho,\varphi) + \frac{1}{\rho} D_\rho^{\alpha} u(\rho,\varphi) + \frac{1}{\rho^2} D_\varphi^{2\alpha} u(\rho,\varphi) = 0 \qquad (1)$$

where $0 < \alpha \leq 1$.

subject to the boundary conditions

$$\begin{aligned} u(1,\varphi) &= u_1, \quad 0 < \varphi < \pi \\ u(1,\varphi) &= u_2, \quad \pi < \varphi < 2\pi \end{aligned} \qquad (2)$$

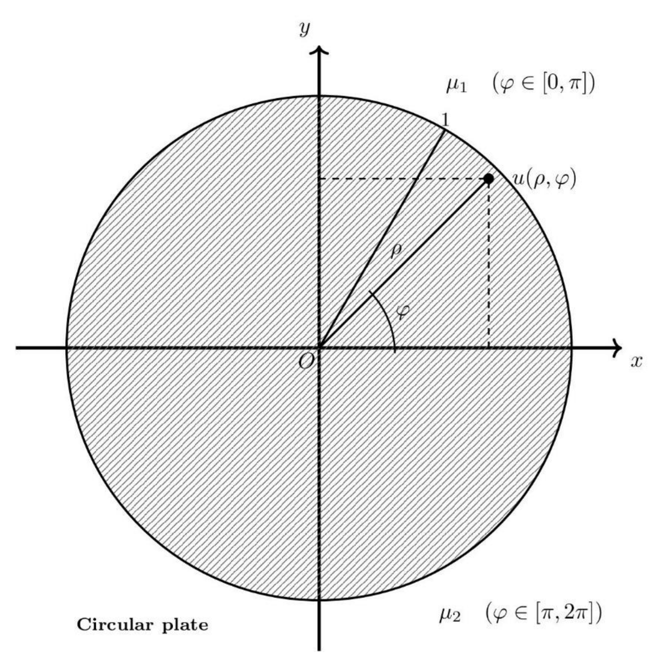


Figure 1: Unit-radius circular plate with insulated faces and prescribed temperatures u_1 (upper boundary) and u_2 (lower boundary).

METHOD

The methodology employed is characterized by its analytical nature and, above all, its didactic approach, and is represented by Figure 2.

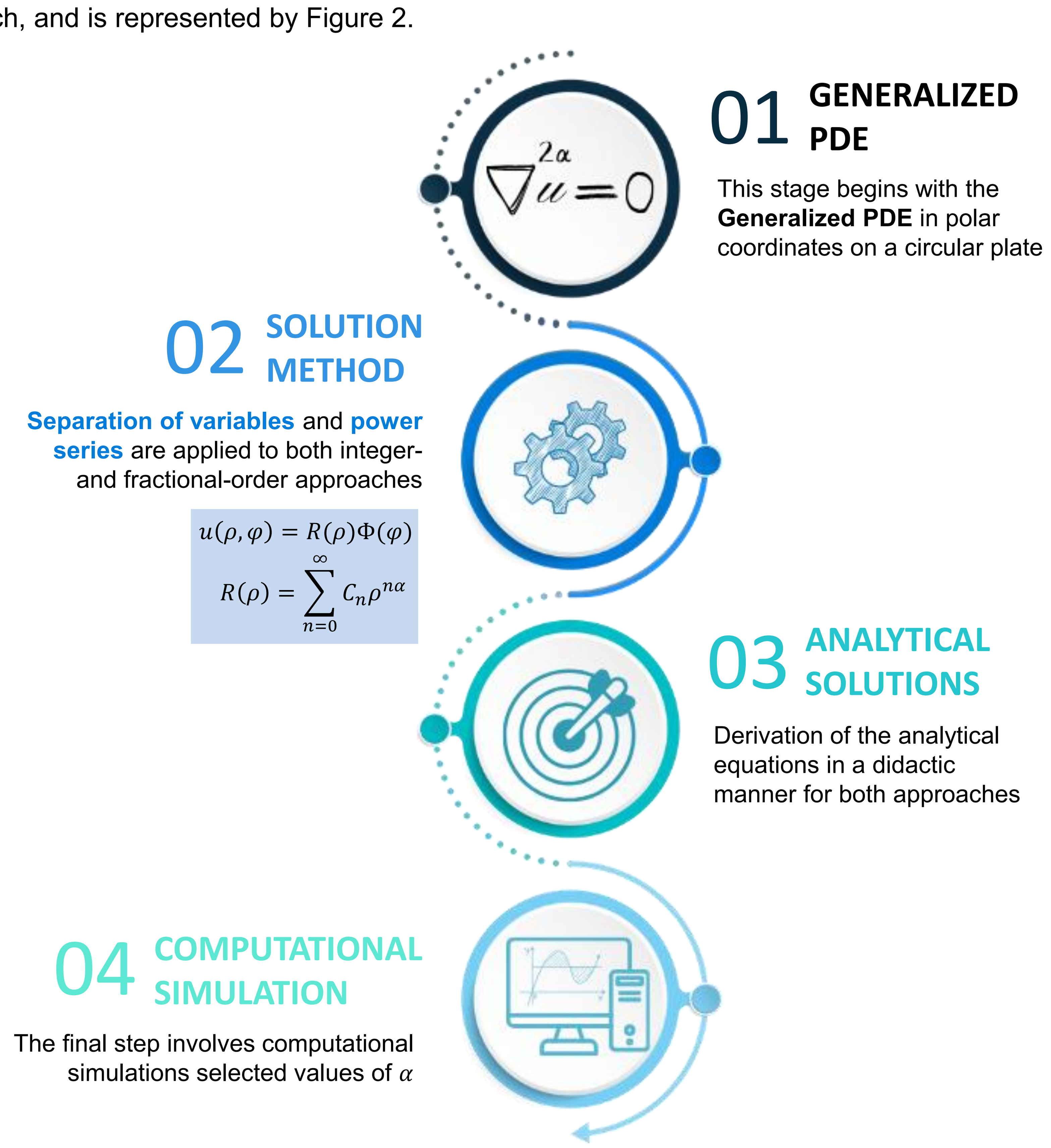


Figure 2: Schematic representation of the methodology employed.

RESULTS & DISCUSSION

The integer-order analytical solution is described as:

$$u(\rho,\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n [a_n \cos(n\varphi) + b_n \sin(n\varphi)] \qquad (3)$$

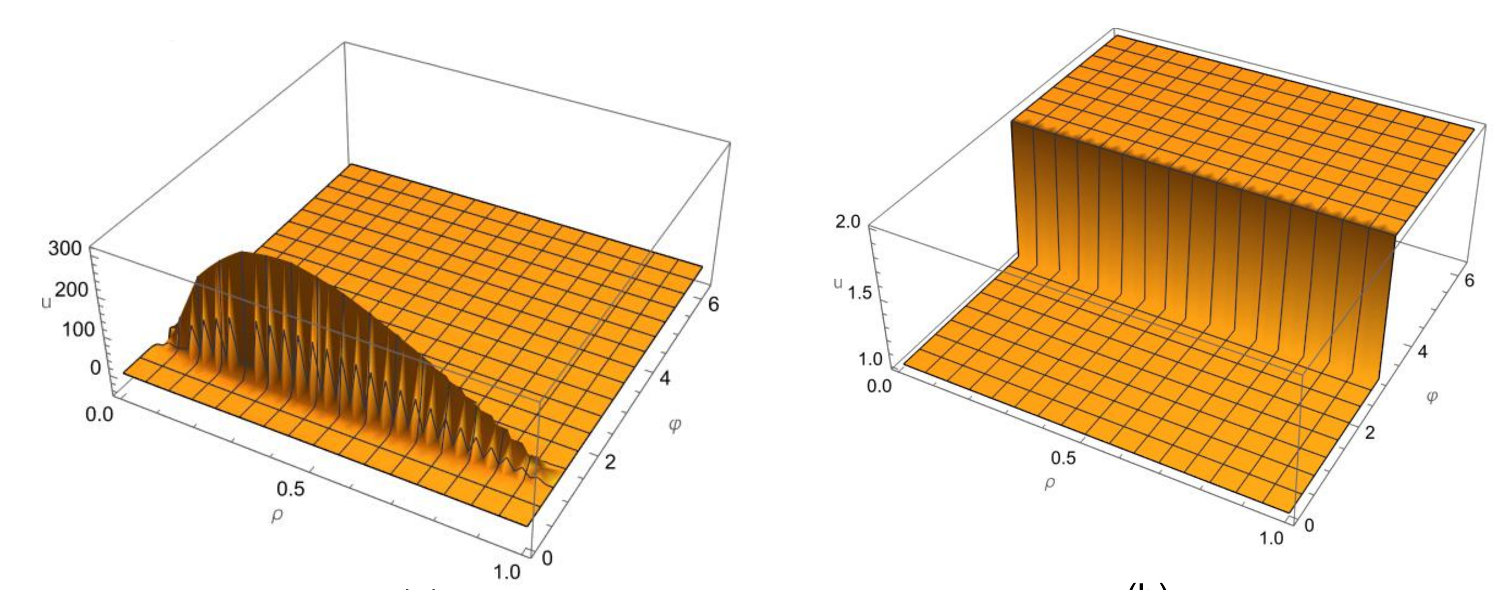
In the fractional case, the recurrence relation depends on the value of α, such that $n \geq \frac{2}{\alpha}$

$$C_{n-\frac{2}{\alpha}+2} = -\frac{C_{n-\frac{1}{\alpha}+1}\Gamma[(n+1)\alpha]\Gamma(n\alpha-1)}{\Gamma(n\alpha)\Gamma[(n+2)\alpha-1]} - \lambda \frac{C_n \Gamma(n\alpha-1)}{\Gamma[(n+2)\alpha-1]} \qquad (4)$$

By setting $\alpha = \frac{1}{2}$, the general fractional-order solution is obtained. Considering $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$, the result is:

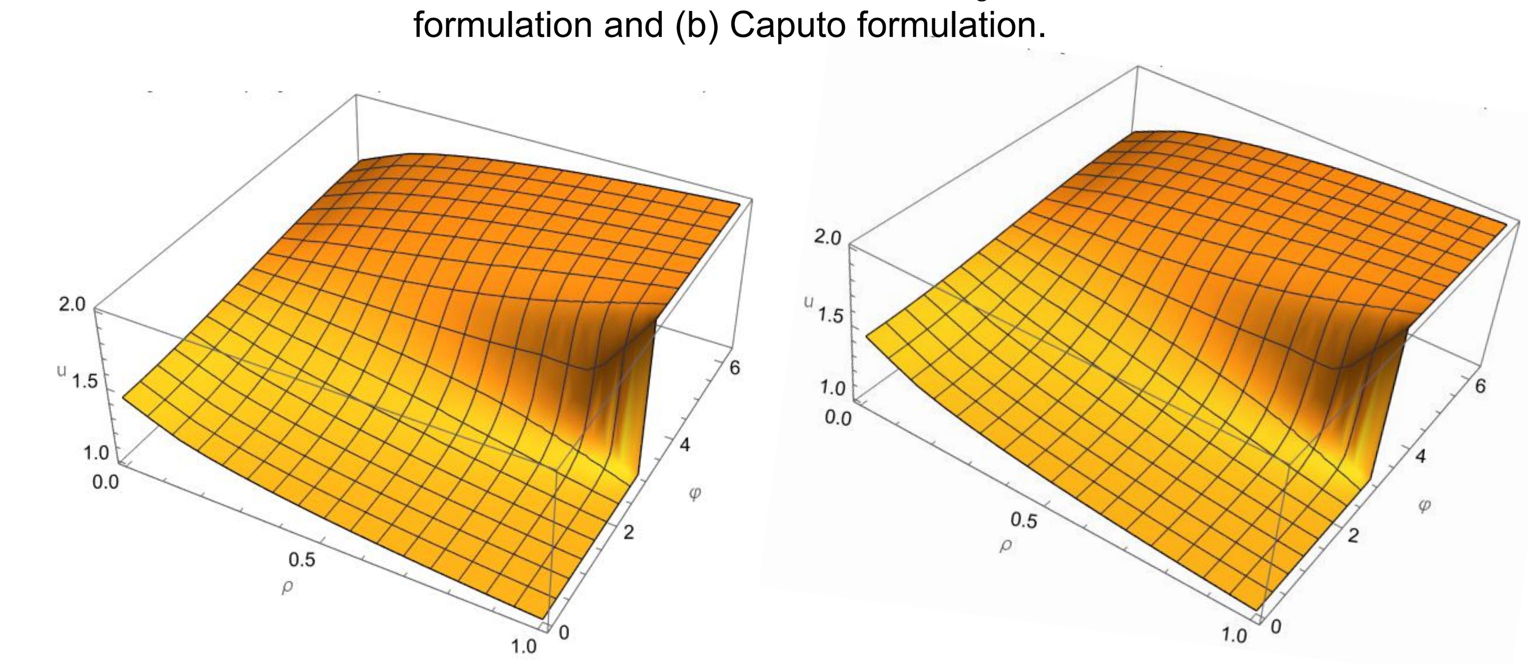
$$u(\rho,\varphi) = A_0[\cos(n\varphi) + i\sin(n\varphi)] \times \left\{ \sum_{k=2}^{\infty} \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} \left[(-1)C_{k+1} \frac{\Gamma\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} + (-1)\lambda C_{k+2} \right] \rho^{\frac{k}{2}} \right\} \qquad (5)$$

The fractional formulation introduces a dependence on the parameter α, such that different values result in distinct solution behaviors.



(a) (b)

Figure 3: Comparison of fractional derivatives for $\alpha = \frac{1}{8}$: (a) Riemann–Liouville formulation and (b) Caputo formulation.



(a) (b)

Figure 4: Comparison of the (a) Riemann–Liouville and (b) Caputo formulations for $\alpha = 1$, recovering the classical derivative.

For $\alpha = 1$, both the Riemann–Liouville and Caputo formulations recover the classical case, while for $\alpha = \frac{1}{8}$, differences between the formulations are observed.

CONCLUSION / FUTURE WORKS

- This work introduces a didactic approach that integrates fractional calculus, separation of variables, and fractional power series methods to model steady-state heat conduction, offering enhanced flexibility and deeper analytical insight;
- Work in progress: study of steady-state heat flow, described by the generalized PDE equation, in an annular domain.

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