IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

MDPI

Free-energy structure of the entropy–extropy ratio R^* on the probability simplex

Arnulfo Dominguez-Hernandez[1], Elisa Dominguez-Hernandez[2], Martha E. Domínguez-Hernandez[3]

1 Graduate Programme in Systems Engineering, IPN, Mexico.

2 Department of Mathematics, Faculty of Higher Studies Cuautitlán, UNAM, Mexico.

3 Department of Agricultural Sciences, Faculty of Higher Studies Cuautitlán, Mexico.

INTRODUCTION & RESEARCH QUESTION

Entropy $H = -\sum p_i \log p_i$ measures uncertainty.

Extropy $J = -\sum(1-p_i)\log(1-p_i)$ measures dispersion in the complementary space.

Normalized ratio:

$$R^*(\mathbf{p}) = \frac{J(\mathbf{p})}{H(\mathbf{p}) + J(\mathbf{p})},$$

Question: Does R^* induce a thermodynamic gradient flow?

CORE THEORY

Free-Energy Functional

Define:

$$\lambda(\mathbf{p}) = \frac{R^*(\mathbf{p})}{1 - R^*(\mathbf{p})}, F(\mathbf{p}) = H(\mathbf{p}) + \lambda(\mathbf{p})\, J(\mathbf{p})$$

Explicit form:

$$\boxed{F(\mathbf{p}) = H(\mathbf{p})\left[1 + \left(\frac{R^*(\mathbf{p})}{1 - R^*(\mathbf{p})}\right)^2\right]}$$

Replicator Dynamics from R^*

Using the Fisher–Shahshahani metric $g_{ij} = \delta_{ij}/p_i$, the gradient flow of R^* yields:

$$\dot{p}_i = p_i(f_i - \bar{f}),$$

$$\bar{f} = \sum_j p_j f_j$$

with fitness:

$$\boxed{f_i = -\frac{[\mathrm{H}\log(1-p_i) + J\log p_i]}{(H+j)^2}}$$

Properties:

•Uniform distribution $\boldsymbol{u} = (\frac{1}{n}, \dots, \frac{1}{n})$ is the unique equilibrium.

•The dynamics is a replicator, a classical model in evolutionary game theory.

GRADIENT FLOWS & ASYMPTOTICS

Gradient Flows

Exact gradient of F:

$$\partial_i F = \partial_i H + \lambda\, \partial_i J + J\, \partial_i \lambda$$

where $\partial_i \lambda = \partial_i R^* / (1 - R^*)^2 \neq 0$.

The replicator dynamics is a valid dissipative system. The functional F has the explicit form presented earlier, but its gradient flow does not coincide with that of R^*. The two are distinct dynamical systems.

REFERENCES

1. Auconi, A. (2025). *Phys. Rev. Lett.* **134**(8), 87104.
2. Harper, M. (2009). *Information Geometry and Evolutionary Game Theory*. arXiv:0911.1383.
3. Lad, F. et al. (2015). *Stat. Sci.* **30**(1), 40–58.
4. Shahshahani, S. (1979). *Memoirs of the AMS* **17**(211).
5. Shannon, C. E. (1949). *Bell Syst. Tech. J.* **27**, 623–656.

KL Divergence Dissipates

A key property of the replicator dynamics is that the Kullback–Leibler divergence serves as a Lyapunov function.

For uniform distribution $\mathbf{u}$:

$$D_{KL}(\mathbf{p} \parallel \mathbf{u}) = \log n - H(\mathbf{p})$$

Along the R^* replicator flow:

$$\frac{d}{dt} D_{KL} \leq 0$$

KL divergence is the fundamental dissipation function, not F.The replicator dynamics derived from R^* is a gradient flow in information geometry but not a gradient flow of the free energy F. F itself is not guaranteed to decrease.

High-Dimensional Limit

For the uniform distribution $\mathbf{u}_n = (\frac{1}{n}, \dots, \frac{1}{n})$:

$$H(\mathbf{u}_n) = \log n\,, J(\mathbf{u}_n) \to 1\ (n \to \infty)$$

Therefore:

$$R^*(\mathbf{u}_n) \sim \frac{1}{\log n} \longrightarrow 0$$

This is the classical log-law. For large n, the ratio collapses to zero, meaning that entropy completely dominates extropy. In this regime, the system's evolution is governed primarily by pure diversity (entropy), and the complementary structure captured by extropy becomes negligible.

In high-dimensional spaces ($n \gg 1$):

- The replicator dynamics derived from R^* becomes increasingly dominated by the entropy term $H\log(1-p_i)$.
- The extropy term $J\log p_i$ contributes only weakly (since J remains $O(1)$ while $H \sim \log n$)
- The uniform distribution remains the unique equilibrium, but relaxation times may scale with $\log n$.

This suggests that for large n, the system behaves approximately like a pure entropy gradient flow, recovering classical results in information geometry.

OPEN QUESTIONS

1. Gradient flow of F: What are its properties? Does it converge to the uniform distribution?
2. Relation between flows: Can one find a transformation making the flows equivalent?
3. Quasi-stationary approximation: Under what precise conditions is $\partial_i \lambda$ negligible?
4. Applications: The replicator may be relevant for eco-evolutionary systems, nonequilibrium thermodynamics, and machine learning.
5. Generalizations: Can R^* be extended using Rényi or Tsallis entropies?

FINAL REMARKS

The entropy–extropy ratio R^* remains an interesting structural measure bridging information geometry, evolutionary dynamics, and thermodynamics. However, its relationship with free energy is more subtle than originally claimed. The explicit form of F and the non-equivalence of gradient flows should guide future research.

ACKNOWLEDGEMENTS

This work was supported by UNAM—PAPIIT IA207626, SECIHTI grant no. 1152947, as well as FES Cuautitlán UNAM Project: CI26107.