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Survival Analysis
Reliability
Modelling

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A Study on the Development of Defective Gompertz-G Family of Distributions

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INTRODUCTION & MOTIVATION

Classical lifetime models assume **all units eventually fail**. Real-world data regularly exhibit:

- ✦ Cured patients
- ✦ Immune individuals
- ✦ Non-failing systems
- ✦ Long-term survivors

The Gompertz distribution captures exponentially increasing hazards. The Gompertz-G family enhances flexibility — yet all existing variants are **proper distributions** whose CDFs converge to unity.

Classical Lifetime Models
↓
Assume Eventual Failure (CDF → 1)
↓
Real Data Contains Cure Fraction
↓
⟹ Need: Defective Gompertz-G Family

PROBLEM STATEMENT

Standard models fail to accommodate a non-zero tail probability representing immune or cured fractions:

MODEL	KEY LIMITATION
Exponential	Constant hazard; CDF → 1
Weibull	No cure fraction; always proper
Gompertz	CDF → 1; no survival plateau
Gompertz-G	Extended yet remains proper
Cure-rate models	Extra parameters; less parsimonious

Defective distributions naturally embed the cure fraction — no additional model components required.

OBJECTIVES

1. Develop the Defective Gompertz-G family
2. Derive PDF, CDF, survival and hazard functions
3. Study key statistical properties
4. Estimate parameters via MLE and Bayesian MCMC
5. Conduct Monte Carlo simulation studies
6. Apply model to real-life censored datasets

PROPER VS DEFECTIVE SURVIVAL

Defective S(x) · Proper S(x) · Cure fraction p_0

METHODOLOGY: DGG CONSTRUCTION (T-X FRAMEWORK)

Baseline G(x) — Any distribution ⊕ **Defective Gompertz** — Generator
↓
Defective Gompertz-G Family (DGG) — Arbitrary baseline · Embedded cure fraction

The **T-X transformation** (Alzaatreh et al., 2013) is adopted. The defective Gompertz generator replaces the classical Gompertz, yielding a family whose survival function plateaus above zero.

PROPOSED DISTRIBUTIONS

Cumulative Distribution Function

$$F(x) = 1 - \exp\left[\frac{\theta}{\gamma}\left\{1 - (1-G(x))^{-\gamma}\right\}\right]$$

Probability Density Function

$$f(x) = \theta\, g(x)(1-G(x))^{-\gamma-1} \exp\left[\frac{\theta}{\gamma}\left\{1-(1-G(x))^{-\gamma}\right\}\right]$$

Survival Function

$$S(x) = \exp\left[\frac{\theta}{\gamma}\left\{1 - (1-G(x))^{-\gamma}\right\}\right]$$

Hazard Function

$$h(x) = \theta\, g(x)\,(1-G(x))^{-\gamma-1}$$

Cure Fraction

$$p_0 = \lim_{x\to\infty} S(x) = \exp\left(\frac{\theta}{\gamma}\right) > 0$$

MODEL VALIDITY

THEOREM — DEFECTIVENESS OF F(X)

- ► $0 < F(x) < 1$ for all finite x in support
- ► $\lim_{x\to x_0} F(x) = 0$ at the lower boundary
- ► $\lim_{x\to\infty} F(x) < 1$ (strictly less than unity)
- ► $\int f(x)\,dx = 1 - \exp(\theta/\gamma) < 1$

"The total probability mass is less than one; the deficit exp(θ/γ) represents the long-term cure or non-failure fraction."

HAZARD FLEXIBILITY

Increasing · Decreasing · Bathtub · Unimodal

STATISTICAL FEATURES

- **CURE FRACTION** — $p_0 = \exp(\theta/\gamma)$ embedded in parameters
- **HAZARD SHAPES** — Increasing, decreasing, bathtub, unimodal
- **QUANTILE FN** — Q(u) via inverse CDF; median = Q(½)
- **MOMENTS / MGF** — μ'_r and M(t) derivable by series expansion
- **ENTROPY** — Rényi & Shannon entropy measures
- **ORDER STATISTICS** — $f_{(k:n)}(x)$ from standard theory

PARAMETER ESTIMATION

① Observed survival / censored data
② Log-likelihood $\ell(\theta, \gamma \mid \text{data})$
③ Score equations & Newton-Raphson (MLE)
④ Bayesian MCMC (Metropolis-Hastings)
⑤ Fisher Information → asymptotic CIs
⑥ AIC / BIC model comparison

MONTE CARLO SIMULATION

RMSE decreases with sample size, confirming estimator consistency.

Bias→0	RMSE↓	CP≈95%	Consist.
asymp.	n ↑	coverage	✓

APPLICATIONS

- Cancer survival analysis
- Reliability engineering
- Biomedical studies
- Epidemiology
- Actuarial science
- Censored survival data

KEY CONTRIBUTIONS

- New defective generator extending the Gompertz-G family
- Flexible hazard structures: increasing, decreasing, bathtub, unimodal
- Cure fraction embedded — no additional cure parameter required
- Unified defective modelling framework for arbitrary baseline distributions
- Directly applicable to censored survival and reliability data

CONCLUSION

The **Defective Gompertz-G (DGG)** family extends the classical Gompertz-G class to accommodate long-term survivors and cure fractions.

The model provides **full hazard flexibility**, parsimonious parameterisation, and is directly applicable to censored survival and reliability datasets across diverse fields.

FUTURE DIRECTIONS

- **REGRESSION** — Covariate-linked DGG regression models
- **MULTIVARIATE** — Defective multivariate extensions
- **BAYESIAN HIER.** — MCMC hierarchical structures
- **ML INTEGRATION** — Neural network cure modelling

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