

A Study on the Development of Defective Gompertz-G Family of Distributions

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INTRODUCTION & MOTIVATION

Classical lifetime models assume all units eventually fail. Real-world data regularly exhibit:

- Cured patients
- Immune individuals
- Non-failing systems
- Long-term survivors

The Gompertz distribution captures exponentially increasing hazards. The Gompertz-G family enhances flexibility — yet all existing variants are **proper distributions** whose CDFs converge to unity.

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graph TD
    A[Classical Lifetime Models] --> B[Assume Eventual Failure CDF = 1]
    B --> C[Real Data Contains Cure Fraction]
    C --> D[=> Need: Defective Gompertz-G Family]
  
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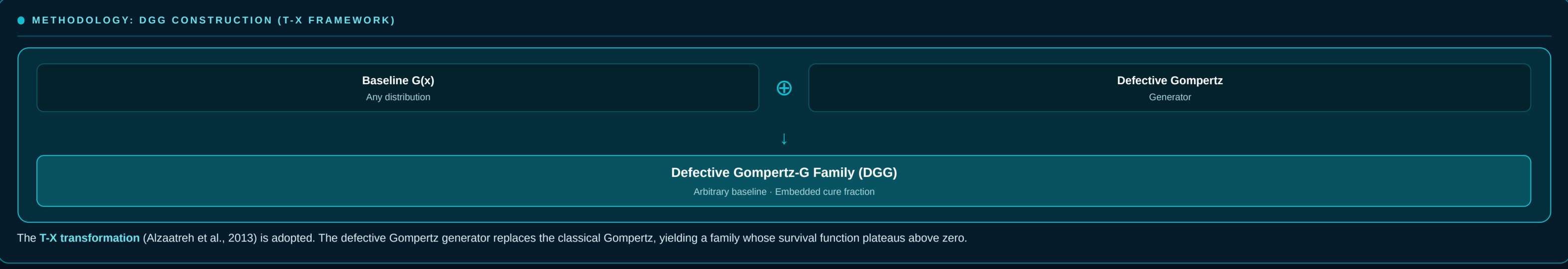
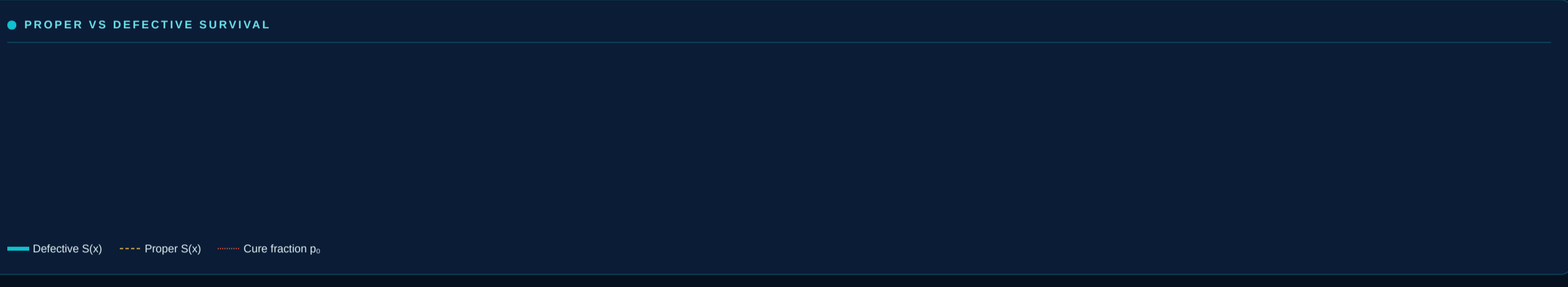
PROBLEM STATEMENT

Standard models fail to accommodate a non-zero tail probability representing immune or cured fractions.

MODEL	KEY LIMITATION
Exponential	Constant hazard; CDF → 1
Weibull	No cure fraction; always proper
Gompertz	CDF → 1; no survival plateau
Gompertz-G	Extended yet remains proper
Cure-rate models	Extra parameters; less parsimonious

Defective distributions naturally embed the cure fraction — no additional model components required.

- ### OBJECTIVES
- 1 Develop the Defective Gompertz-G family
 - 2 Derive PDF, CDF, survival and hazard functions
 - 3 Study key statistical properties
 - 4 Estimate parameters via MLE and Bayesian MCMC
 - 5 Conduct Monte Carlo simulation studies
 - 6 Apply model to real-life censored datasets



PROPOSED DISTRIBUTIONS

Cumulative Distribution Function

$$F(x) = 1 - \exp\left\{-\frac{\theta}{\gamma} \left[1 - (1-G(x))^\gamma\right]\right\}$$

Probability Density Function

$$f(x) = \theta g(x) [1-G(x)]^{\gamma-1} \exp\left\{-\frac{\theta}{\gamma} \left[1 - (1-G(x))^\gamma\right]\right\}$$

Survival Function

$$S(x) = \exp\left\{-\frac{\theta}{\gamma} \left[1 - (1-G(x))^\gamma\right]\right\}$$

Hazard Function

$$h(x) = \theta g(x) [1-G(x)]^{\gamma-1}$$

Cure Fraction

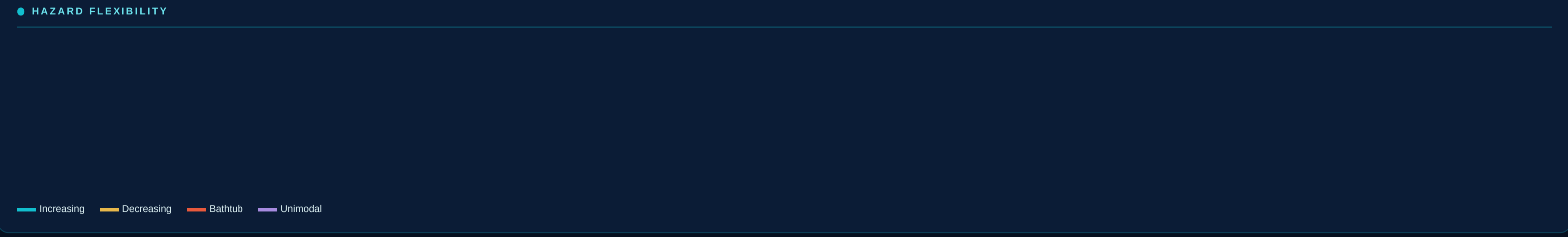
$$p_0 = \lim_{x \rightarrow \infty} S(x) = \exp\left(-\frac{\theta}{\gamma}\right) > 0$$

MODEL VALIDITY

THEOREM — DEFECTIVENESS OF F(x)

- ▶ $0 < F(x) < 1$ for all finite x in support
- ▶ $\lim_{x \rightarrow \infty} F(x) = 0$ at the lower boundary
- ▶ $\lim_{x \rightarrow \infty} F(x) < 1$ (strictly less than unity)
- ▶ $\int f(x) dx = 1 - \exp(-\theta/\gamma) < 1$

"The total probability mass is less than one; the deficit $\exp(-\theta/\gamma)$ represents the long-term cure or non-failure fraction."



STATISTICAL FEATURES

CURE FRACTION $p_0 = \exp(-\theta/\gamma)$ embedded in parameters	HAZARD SHAPES Increasing, decreasing, bathtub, unimodal
QUANTILE FN $Q(p)$ via inverse CDF; median = $Q(0.5)$	MOMENTS / MGF μ_r and $M(t)$ derivable by series expansion
ENTROPY Rényi & Shannon entropy measures	ORDER STATISTICS $t_{(k:n)}$ from standard theory

- ### PARAMETER ESTIMATION
- 1 Observed survival / censored data
 - 2 Log-likelihood $\ell(\theta, \gamma | \text{data})$
 - 3 Score equations & Newton-Raphson (MLE)
 - 4 Bayesian MCMC (Metropolis-Hastings)
 - 5 Fisher information — asymptotic CIs
 - 6 AIC / BIC model comparison

Monte Carlo Simulation

RMSE decreases with sample size, confirming estimator consistency.

Bias: -0.0015 semp.	RMSE: 0.11	CP: 95% coverage	Consist. ✓
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APPLICATIONS

- Cancer survival analysis
- Reliability engineering
- Biomedical studies
- Epidemiology
- Actuarial science
- Censored survival data

KEY CONTRIBUTIONS

- New defective generator extending the Gompertz-G family
- Flexible hazard structures: increasing, decreasing, bathtub, unimodal
- Cure fraction embedded — no additional cure parameter required
- Unified defective modeling framework for arbitrary baseline distributions
- Directly applicable to censored survival and reliability data

CONCLUSION

The Defective Gompertz-G (DGG) family extends the classical Gompertz-G class to accommodate long-term survivors and cure fractions. The model provides full hazard flexibility, parsimonious parameterisation, and is directly applicable to censored survival and reliability datasets across diverse fields.

FUTURE DIRECTIONS

REGRESSION Covariate-linked DGG regression models	MULTIVARIATE Defective multivariate extensions
BAYESIAN INF. MCMC hierarchical structures	ML INTEGRATION Neural network cure modeling

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