IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

MDPI

# New Parametric Curves for the Brachistochrone Optimal Control Problem Using the Dynamic Programming Method

Amroune Nasreddine, Ghanem Aicha

National Higher School of Mathematics (NHSM), Scientific and Technology Hub of Sidi Abdellah, P.O. Box 75, Algiers 16093, Algeria

## INTRODUCTION & AIM

The Brachistochrone problem, initially proposed by John Bernoulli in 1696 as a challenge to the mathematical community, holds great significance in the development of calculus of variations. John Bernoulli himself solved the problem several months later, followed by his brother Jacob Bernoulli, as well as other eminent mathematicians like Isaac Newton, Gottfried Leibniz, and Leonhard Euler. This problem addresses finding the optimal curve for a frictionless slide of an object under the influence of a uniform gravitational force, with the goal of minimizing travel time. St. Mirica provided a theoretical solution to this problem using dynamic programming in [3], without explicitly representing the curves. The present work aims to present new comparative parametric curves of the Brachistochrone.

## Mirica's Dynamic programming formulation

Given $k = \frac{(v_0)^2}{2g} \geq 0$, for any $y = (y_1, y_2) \in Y_0$. Find:

$$\inf C(x(.)) = \int_0^{t_1} \frac{\|x'(t)\|}{\sqrt{x_2(t)+k}} dt, \ \|v\| = \sqrt{(v_1)^2 + (v_2)^2},$$

subject to:

$$\begin{aligned} & x'(t) \in F(x(t)) \text{ a.e } ([0,t_1]), \\ & x(.) \in \Omega_1(y) = \{x(.) \in AC; x(0) = y,\ x(t_1) = 0\} \\ & x(t) \in Y_0 \ \forall t \in [0,t_1),\ x(t_1) \in Y_1, \end{aligned}$$

The problem in 1 is a particular example in the more general class of Lagrange autonomous optimal control problems, defined by the following data:

$$\begin{aligned} & Y_0 = (0,+\infty) \times (-k,+\infty),\ Y_1 = \{(0,0)\},\ g(0,0) = 0, \\ & F(x) = \mathbb{R}^2 \ \forall x \in Y = Y_0 \cup Y_1,\ g_0(x,v) = \frac{\|v\|}{\sqrt{x_2+k}},\ \Omega_a = \Omega_1 = AC. \end{aligned}$$

### Characterization of the Hamiltonian and the set of extremal points:

The pseudo-Hamiltonian is given in our case by H(x, p, u, v) = ⟨p, v⟩ + ‖v‖/√(x2+k) .

**Proposition:** The Hamilonian and the corresponding marginl multifunction are given by tl

$$\begin{aligned} H(x,p) &= \begin{cases} 0 & if\ \|p\| \leq 0, \\ -\infty & if\ \|p\| > 0, \end{cases} \\ \hat{F}(x,p) &= \begin{cases} \{(0,0)\} & if\ \|p\| < \frac{1}{\sqrt{x_2+k}} = N(x_2), \\ \emptyset & if\ \|p\| > N(x_2), \\ \{\mu.p; \mu \leq 0\} & if\ \|p\| = N(x_2), \end{cases} \\ Z &= dom(H(x,p)) = \{(x,p) \in Y_0 \times \mathbb{R}^2; \|p\| \leq N(x_2)\}. \end{aligned}$$

The domain Z is finitely-stratified by the stratifcation SH = {S0, S1} defined by:

$$\begin{aligned} S_0 &= \{(x,p) \in Y_0 \times \mathbb{R}^2; \|p\| < N(x_2)\}, \\ S_1 &= \{(x,p) \in Y_0 \times \mathbb{R}^2; \|p\| = N(x_2)\}, \end{aligned}$$

If we denote by: H0(., .) = H(., .) | Z0 and H1(., .) = H(., .) | Z1, then it follows: H0(., .) = 0 if (x, p) ∈ S0 and H1(., .) = 0 if (x, p) ∈ S1.

**Generalized Hamiltonian and characteristic flow:** The Hamiltonian oriental field of H0,1(., .) is defined by:

$$d^{\#}H_{0,1}(x,p) = \Big\{(x',p') \in T_{(x,p)}S_{0,1}; x' \in \hat{F}(x,p),\ <x',v> - <p',u> = DH_{0,1}(x,p).(u,v)\ \forall (u,v) \in T_{(x,p)}S_{0,1},\ (x,p) \in S_{0,1}\Big\},$$

where T(x,p)S denotes the tangent space to S0,1 at the point (x, p) ∈ S0,1, if S0,1 ⊂ Rn × Rn is an open sub-manifold then the Hamiltonian orientor field coincides with the classical Hamiltonian vector filed:

$$d^{\#}H_{S_{0,1}}(x,p) = \Big\{\Big(\frac{\partial H_{0,1}}{\partial p}(x,p), -\frac{\partial H_{0,1}}{\partial x}(x,p)\Big)\Big\} \forall (x,p) \in S_{0,1},$$

S0 is open and H(x, p) = 0 ∀(x, p) ∈ S0 from the formula in (8), one obtains:

$$d_s^{\#}H(x,p) = \Big\{(0,0) \subset \mathbb{R}^2 \times \mathbb{R}^2 \ \forall (x,p) \in S_0\Big\},$$

next, the orientor filed on the 3-dimensional manifold S1 has been already computed to give:

$$d_s^{\#}H_1(x,p) = \{(x',p') \in \mathbb{R}^2 \times \mathbb{R}^2; x' = -2p_2'(x_2+k)^2.p,\ p' = (0,p_2'),\ p_2' \geq 0\},$$

**Remark**: On the open stratum S0 the corresponding differential system (x′, p′) = (0, 0 produce only constant trajectories.

## RESULTS & DISCUSSION

**The Hamiltonian system on the stratum S1:**

The Hamiltonian system on the stratum S1 (for wish ‖p‖ = √ (1x2+k) = N(x2)):

$$\begin{cases} x_1' = -2x_2, & x_1(0) = 0, \\ x_2' = -2x_2\frac{p_2}{p_1}, & x_2(0) = 0, \\ p_1' = 0, & p_1(0) = q_1 \in (0,\infty), \\ p_2' = \frac{(p_1^2+p_2^2)}{p_1}, \end{cases}$$

keeping only the solutions *(X(., θ), P(., θ)) that satisfy the conditions:*

$$X(t,\theta) \in Y_0 = (0,\infty) \times (0,\infty)\ \forall t \in I_0(\theta) = (t^-(\theta),0).$$

The solution of system in the form of maximal flows is given by the formulas:

$$\begin{cases} X_1(t,\lambda) = \frac{-2t+\sin 2t}{2\lambda^2}, t \in I(\lambda) = (-\pi,0], \\ X_2(t,\lambda) = \frac{\sin^2 t}{\lambda^2}, \\ P_1(t,\lambda) = \lambda, \\ P_2(t,\lambda) = -\lambda\frac{\cos t}{\sin t}, \forall (t,\lambda) \in B_0 = (-\pi,0) \times (0,\infty). \end{cases}$$

**Numerical results :** We present several examples to illustrate the efficiency of the dynamic model in the representation of trajectories. For the implementation, we use MATLAB R2010a executed on a Core i7.

**Example 1 :**

We take the values of t as arithmetic sequence, with *λ ∈ (0,+∞), also t ∈ (−π, 0):*

$$t = -\pi + 10^{-3} + \frac{n}{10},\ \forall n \in [-10^{-2}, 10\pi - 10^{-2}),$$

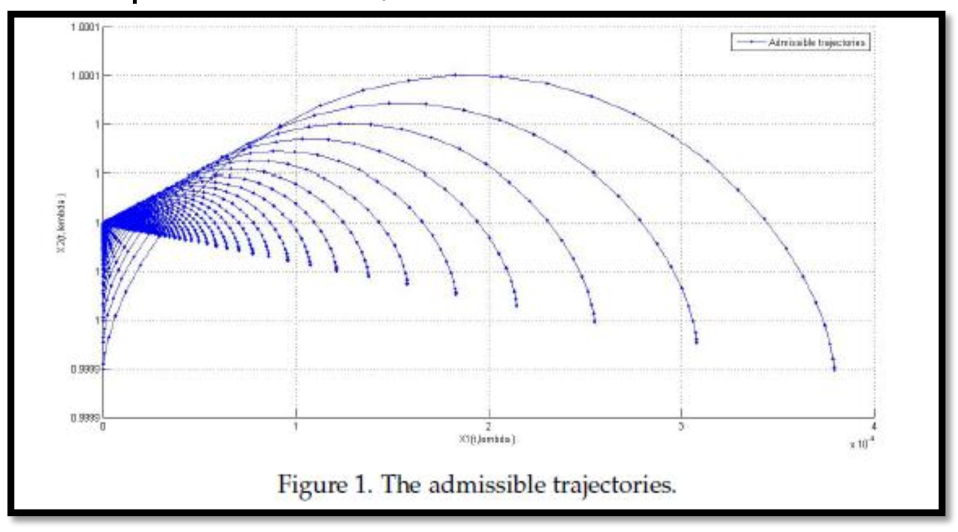

Figure 1. The admissible trajectories.

**Example 2 :**

We take the values of t as arithmetic sequence, with *λ ∈ (0,+∞), also t ∈ (−π, 0):*

$$t = -\pi + 10^{-3} + \frac{n}{100},\ \forall n \in [-10^{-1}, 100\pi - 10^{-1}),$$

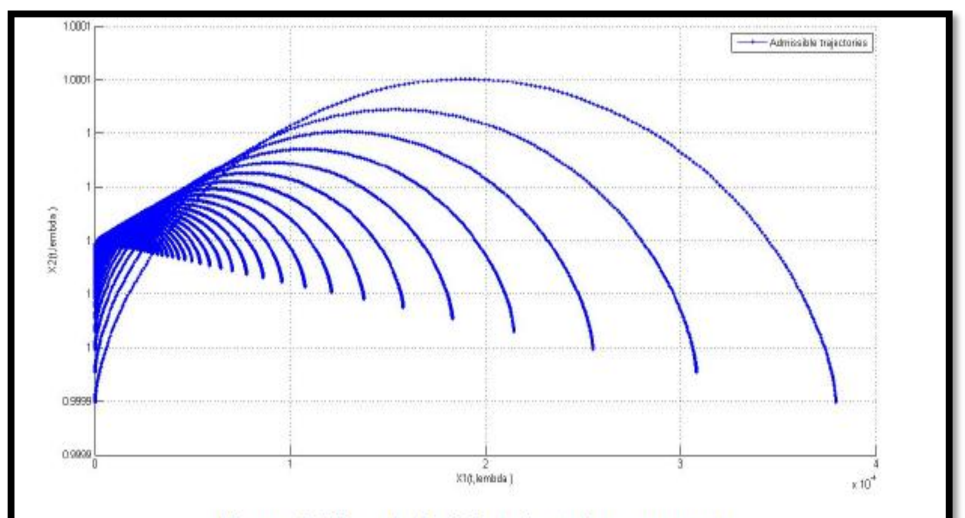

Figure 2. The admissible trajectories zoom out.

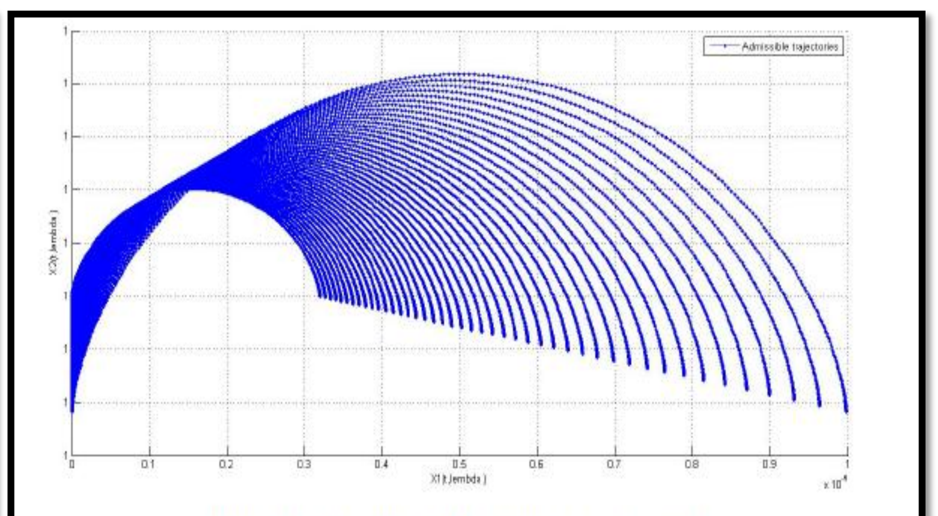

Figure 3. The admissible trajectories Zoom In.

## CONCLUSION

The parametric representation of the Brachistochrone curve demonstrates that the travel time, the curve value and the velocity along the trajectory can be computed and systematically compared for each curve. Moreover, even a very small variation in the parameter t significantly affects the curve value, the traveled distance and the velocity profile along the curve. This behavior arises from the fact that curves that do not originate from the same height do not yield identical velocities throughout the motion.

## REFERENCES

1. Bernoulli, J. Problema novum ad cujus solutionem mathematici invitantur. Acta Eruditorum 1697, 269–273. 102
2. Bellman, R. Dynamic Programming; Princeton University Press: Princeton, NJ, USA, 1957.
3. Mirica, S. A Dynamic Programming Solution to Bernoulli's Brachistochrone Problem. Rev. Roumaine Math. 1996, 87, 221–242.
4. Mirica, S. Constructive Dynamic Programming in Optimal Control; Editura Academiei: Bucuresti, Romania, 2004.
5. Mirica, S. User's Guide on Dynamic Programming for Autonomous Differential Games and Optimal Control Problems. Rev. Roumaine Math. 2004, 49, 501–529.
6. Sussmann, H.J.; Willems, J.C. 300 Years of Optimal Control: From the Brachistochrone to the Maximum Principle. IEEE Control Systems Magazine 1997, 17, 32–44.

aicha.ghanem@nhsm.edu.dz

https://sciforum.net/event/iocma2026