IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

MDPI

Homolojical Objects of the Proper Class determined by Weakly D-Closed Modules

Ergül TÜRKMEN, Yılmaz Mehmet DEMİRCİ, Sait TAŞ

Amasya University, Faculty of Science and Arts, Turkey, ergul.turkmen@amasya.edu.tr

Abdullah Gül University, Faculty of Engineering, Turkey, yilmazmdemirci@gmail.com

Atatürk University, Faculty of Science, Turkey, saittas@atauni.edu.tr

INTRODUCTION & AIM

A module *M* is said to **crumble** (or be a **crumbling module**) if socles split in every factor module; that is, Soc(*M/N*) is a direct summand of *M/N* for every submodule *N* ≤ *M*. It follows from [8, Corollary 2] that a module *M* crumbles if and only if it is a locally Noetherian *V*-module.

In [1], the sum of all crumbling submodules of a module (M) is denoted by C(M). By [1, Propositions 3.1 and 3.4], C(M) is the largest crumbling submodule of M, and Soc(M)≤C(M). The authors of [1] also studied modules whose factor modules possess nonzero crumbling submodules. A module M is called *weakly semiartinian* if C(M/N)≠ 0 for every proper submodule N of M. The sum of all weakly semiartinian submodules of M is the largest weakly semiartinian submodule of M, denoted by wsa(M). Clearly, every semiartinian module and every crumbling module is weakly semiartinian. Further properties and characterizations of weakly semiartinian modules can be found in the same work.

Let $\mathbb{T}_A$ *denote the class of all weakly semiartinian modules, and let* $\mathbb{F}_A$ denote the class of all modules M satisfying C(M)=0. By [1, Proposition 2 and Theorem 6], the pair $(\mathbb{T}_A, \mathbb{F}_A)$ forms a hereditary torsion theory. We refer to this torsion theory as the *Ada torsion theory*.

The notion of D-closed submodules was introduced in [2]. A submodule N of a module M is said to be D-closed in M if Soc(M/N) = 0. Motivated by the Ada torsion theory, we introduce the concept of weakly D-closed submodules as a natural generalization of D-closed submodules. A submodule N of a module M is called weakly D-closed in M if M/N belongs to the torsion-free class $\mathbb{F}_A$; equivalently, if C(M/N) = 0. Since Soc(X) ≤ C(X) for every module X, every weakly D-closed submodule is necessarily D-closed.

METHOD

We denote by WD-Closed the class of all short exact sequences

$$0 \to A \xrightarrow{f} B \to C \to 0$$

for which Im(f) is a weakly D-closed submodule of B. The following example shows that, in general, the class WD-Closed does not necessarily form a proper class.

Example. Let N be a module with C(N) ≠0. Consider the short exact sequence

$$0 \to 0 \to N \xrightarrow{i} N \to 0,$$

where the map i:N→N is the identity homomorphism. This sequence is split exact. However, it does not belong to the class weakly D-Closed, since the image of the zero map is 0 and N/0≅ N, while C(N)≠ 0. Hence 0 is not a weakly D-closed submodule of N. Therefore, the class WD-Closed does not contain all split short exact sequences. Consequently, WD-Closed is not a proper class.

In order to obtain a proper class containing WD-Closed, we introduce the notion of extended weakly D-closed submodules (see 6). A submodule N of a module M is called extended weakly D-closed in M if there exists a submodule K≠ M such that K∩N=0) and C(M/(K⊕N))=0. By taking K=0, it is immediate that every weakly D-closed submodule is extended weakly D-closed. However, the converse does not hold in general, as shown in the above Example.

We denote by EWD-Closed the class of all short exact sequences

$$0 \to A \to B \to C \to 0$$

such that the image of the monomorphism Im(f) is an extended weakly D-closed submodule of B. By [6, Theorem 2.1], EWD-Closed is the smallest proper class generated by the class WD-Closed of short exact sequences.

Proposition. The proper class generated by WD-Closed is precisely EWD-Closed. In other words, EWD-Closed is the smallest proper class containing WD-Closed.

Let X be a class of modules. The collection of all short exact sequences $\mathbb{E}$ for which every module in X is projective relative to $\mathbb{E}$ forms a proper class, called the proper class projectively generated by X. We denote by WS the proper class projectively generated by the class of all weakly semiartinian modules.

Theorem. The proper class generated by WD-Closed coincides with both EWD-Closed and WS. In particular, the smallest proper class containing WD-Closed is precisely the proper class projectively generated by the class of all weakly semiartinian modules.

RESULTS & DISCUSSION

Proposition. Every module with zero crumbling submodule is projective if and only if Split=WS. Equivalently, every extended weakly D-closed short exact sequence splits.

Since every simple module is weakly semiartinian, we deduce that WD-Closed ⊆ WS ⊆ Neat. Moreover, Closed ⊆ Neat by [5, Proposition 5]. The converse inclusion holds if and only if R is a right C-ring by [4, Theorem 5]. Therefore, for every right C-ring, we obtain WD-Closed ⊆ Closed.

Proposition. Neat coincides with WS if and only if every weakly semiartinian module K can be written as K = S⊕P, where S is semisimple and P is projective.

Definition. Let M be a module. We say that M is **WS-flat** if every short exact sequence with ending M belongs to the proper class WS. That is, whenever

$$0 \to A \to B \to M \to 0$$

is a short exact sequence, it belongs to WS.

Example. Every crumbling-free module is WS-flat.

Proposition. The following statements are equivalent for a ring R.
1. R is crumbling-free.
2. Every projective left R-module is crumbling-free.
3. Every WS-flat left R-module is crumbling-free.

Proposition. The class of WS-flat modules is closed under direct summands, extensions, and finite direct sums.

Theorem The following statements are equivalent for a ring R.
1. R is semisimple.
2. Every left R-module is WS-flat.
3. Every weakly semiartinian left R-module is WS-flat.

CONCLUSION

In this study, we introduced the proper class WS generated by weakly semiartinian modules and studied its connections with weakly D-closed and neat exact sequences. Furthermore, we introduced the notion of WS-flat modules and established their basic properties, characterizations, and closure conditions. Several module-theoretic and ring-theoretic characterizations related to weakly semiartinian modules and WS-flatness were also obtained.

FUTURE WORK / REFERENCES

1. Alizade, R.; Demirci, Y.M.; Nişancı Türkmen, B.; Türkmen, E. On rings with one middle class of injectivity domains. *Math. Commun.* **2022**, *27*, 109–126.
2. Durğun, Y.; Özdemir, S. On (\mathcal{D})-closed submodules. *Proc. Indian Acad. Sci. Math. Sci.* **2020**, *130*, 1. https://doi.org/10.1007/s12044-019-0537-1.
3. Fuchs, L. Neat submodules over integral domains. *Period. Math. Hung.* **2012**, *64*, 131–143.
4. Generalov, A.I. On weak and high purity in the category of modules. *Math. USSR-Sb.* **1978**, *34*, 345. https://doi.org/10.1070/SM1978v034n03ABEH001209.
5. Stenström, B.T. High submodules and purity. *Ark. Math.* **1967**, *7*, 173–176.
6. Kepka, T. On one class of purities. *Comment. Math. Univ. Carolin.* **1973**, *14*, 139–154.
7. Buchsbaum, D.A. A note on homology in categories. *Ann. Math.* **1959**, *69*, 66–74. https://doi.org/10.2307/1970093.
8. Er, N.; López-Permouth, S.; Sökmez, N. Rings whose modules have a maximal or minimal injectivity domains. *J. Algebra* **2011**, *330*, 404–417.
9. Wisbauer, R. *Foundations of Module and Ring Theory*; Taylor & Francis: London, UK, 1991.