Approximation and Decomposition of Maps in L^2 via Fractal Structures

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Abstract

We introduce a theoretical framework for the approximation and decomposition of functions in $L^2 := L^2([0,1])$ with the usual Lebesgue measure using fractal structures. From a family $\mathcal{B}$ of generating functions we build the spaces V_n obtained by translations and rescalings of $\mathcal{B}$ over the elements of the level of a fractal structure Γ_n. Wavelet decompositions arise as a special case. A central rigidity result characterises universal approximation by a single generator, and the convergence speed is quantified through fractal versions of the Jackson and Bernstein–Walsh theorems. The density results extend to general spaces endowed with a finite Radon measure.

Fractal structures and the spaces V_n

A prefractal structure on a topological space X is a countable family of coverings Γ_n such that for each point $x \in X$, the family $\{X \setminus \bigcup_{x\notin A; A\in\Gamma_n} A : n \in \mathbb{N}\}$ forms a neighbourhood basis for each x. Each covering is also called a level. Each element may be referred to as a cell. A fractal structure on X is a prefractal structure such that for each pair of consecutive levels, Γ_{n+1} is a strong refinement of Γ_n. A usual fractal structure $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ on $[0,1]$ is a sequence of tessellations into closed intervals with Γ_{n+1} a strong refinement of Γ_n and $\operatorname{diam}(\Gamma_n) \to 0$. The natural structure on $[0,1]$ is

$$\Gamma_n = \left\{ \left[\tfrac{k_i}{2^n}, \tfrac{k_i+1}{2^n}\right] : k_i \in [0, 2^n) \cap \mathbb{Z} \right\}.$$

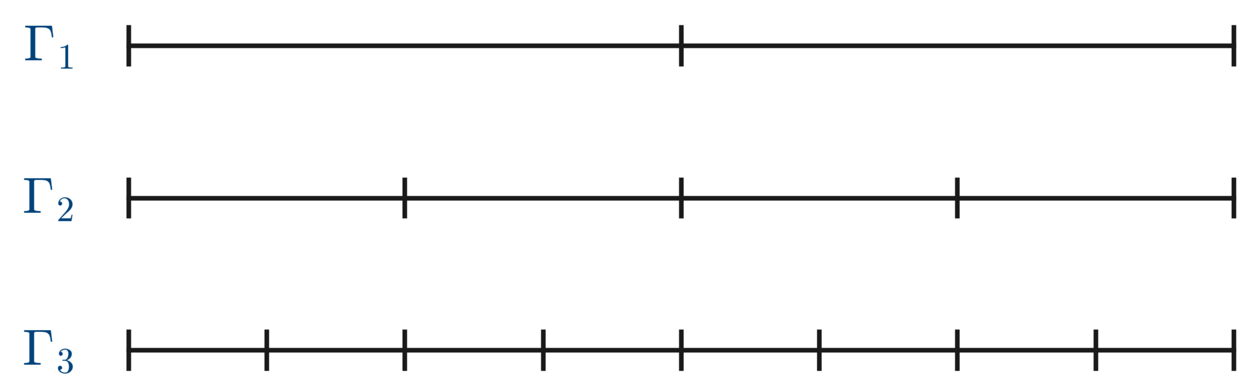

Figure 1. First three levels of the natural fractal structure on $[0,1]$.

A generator $B : [0,1] \to \mathbb{R}$ is an element of L^2. Each cell A is equipped with the affine rescaling $\sigma_A : A \to [0,1]$, $\sigma_A(x) = (x - \min A)/|A|$, and a generator $B \in L^2$ is transported by $B(A)(x) = B(\sigma_A(x))\,\mathbf{1}_A(x)$. Given a family of generators $\mathcal{B} = \{B_i\}_{i\in I}$ (either finite or countably infinite), we define $V_n(\mathbf{\Gamma}, \mathcal{B}) = \operatorname{span}\{\, B_i(A) : A \in \Gamma_n,\ i \in I \,\}$. It may be referred to as V_n when the context is clear.

Three notions are studied: level approximation $d(f, V_n) \to 0$; accumulated approximation $\operatorname{Cl}\big(\sum_n V_n\big) = L^2$; and L^2 decomposition $f = \sum_n g_n$ with $g_n \in V_n$. Note that these notions may be extended to $L^2(\mathbb{R})$; for each $\varepsilon > 0$ it is possible to restrict the target function to a closed interval such that $\|f - f|_{[a,b]}\| \le \varepsilon$.

Rigidity of a single generator

The simplest case is sharply constrained: a single generator fills L^2 only if it is constant. Moreover, the notion of accumulated approximation and decomposition are indistinguishable when just one generator is considered.

Theorem (rigidity)

Let $\mathbf{\Gamma}$ be a usual fractal structure and $B \in L^2$ with $B \not\equiv 0$. Then $(\mathbf{\Gamma}, B)$ provides level approximation of *every* $f \in L^2$ if and only if B is constant almost everywhere.

Mean value governs density

Let $\mathbf{\Gamma}$ be a usual fractal structure. $\int_0^1 B \neq 0 \iff (\mathbf{\Gamma}, B)$ gives accumulated approximation $\iff$ every $f \in L^2$ admits an L^2 decomposition $f = \sum_n g_n$. A generator with zero mean can never, by itself, generate L^2.

Decomposition of a function

Under suitable hypotheses, the decomposition exists and can be constructed directly.

Theorem (decomposition)

Let (V_n) be closed subspaces with $V_n \subseteq V_{n+1}$ and $\operatorname{Cl}\big(\bigcup_n V_n\big) = L^2$. Then every $f \in L^2$ admits a decomposition $f = \sum_{n\ge1} g_n$ with $g_n \in V_n$, obtained from successive residuals $g_n = s_n - s_{n-1}$ where s_n is the accumulated approximation up to a certain level.

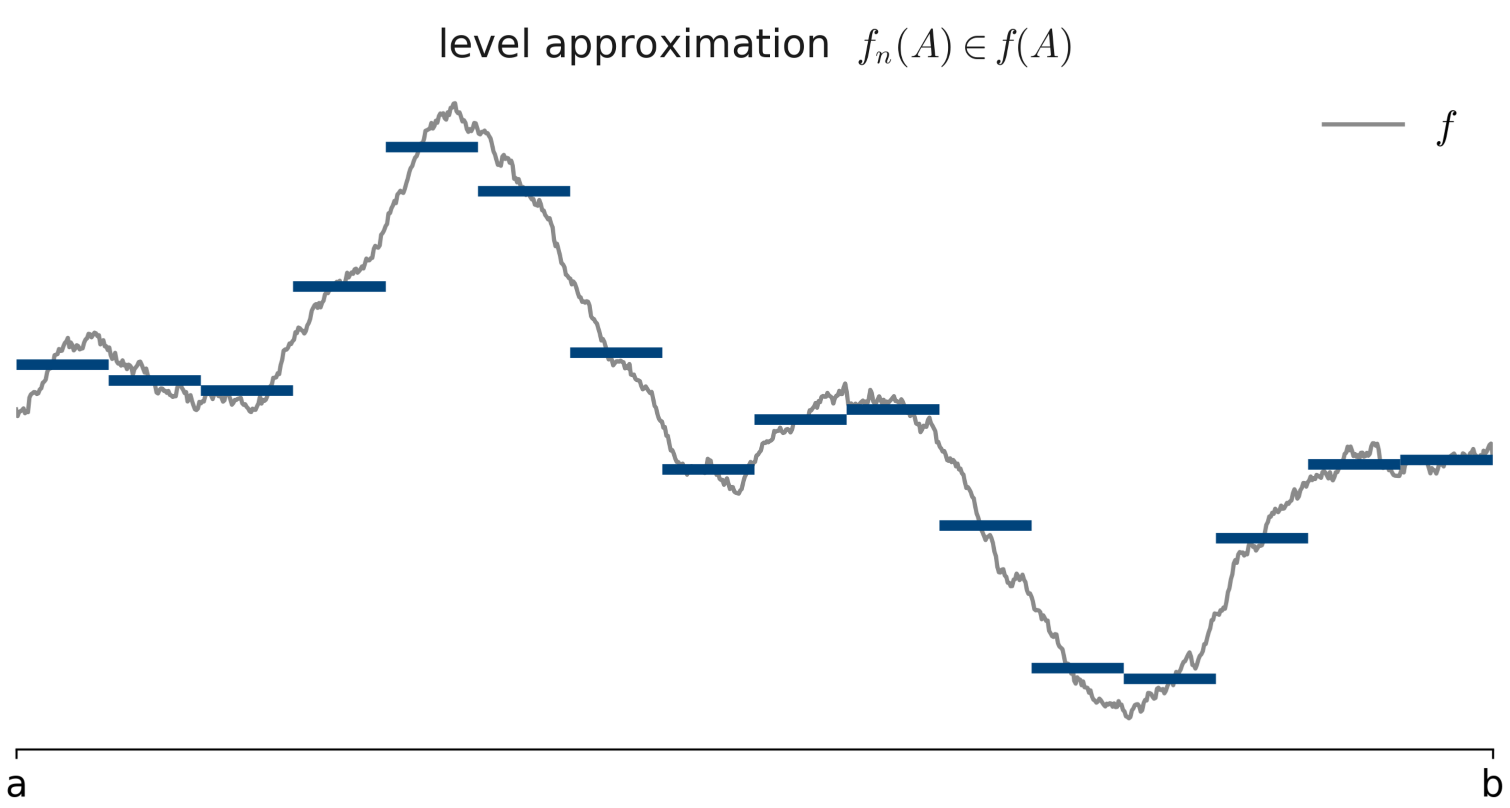

Figure 2. Level approximation of an irregular signal $f : [a,b] \to \mathbb{R}$: each blue mark is a value $f_n(A) \in f(A)$.

Orthogonal decomposition

Orthogonality and completeness induce some interesting results.

Theorem (one function is not enough)

If $\{B(A) : A \in \bigcup_n \Gamma_n\}$ is orthogonal in L^2 then $B = 0$ a.e. Hence an orthogonal L^2 decomposition needs at least two functions: a scaling φ with $\int \varphi \neq 0$ and a wavelet ψ with $\int \psi = 0$. Countably infinite families of functions non-zero almost everywhere are likewise impossible.

Theorem (number of children)

If a finite family of generators yields an orthogonal L^2 decomposition with nested spaces $V_n \subseteq V_{n+1}$, then each cell has exactly k children and the first level has exactly k elements.

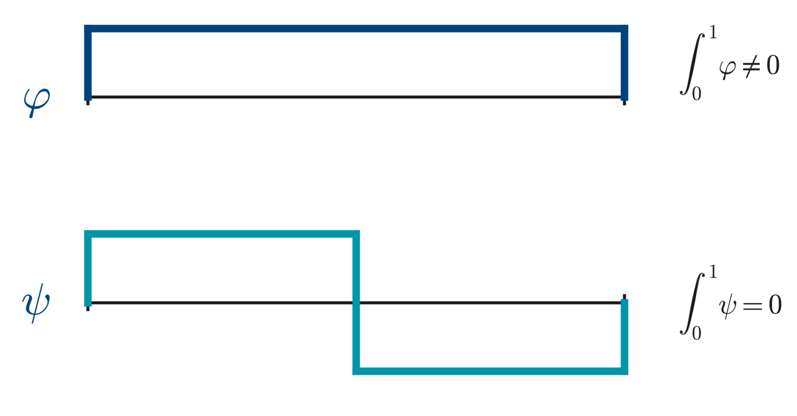

Figure 3. Scaling function φ ($\int \varphi \neq 0$) and wavelet ψ ($\int \psi = 0$): the building blocks of the orthogonal decomposition.

Speed of approximation

Extending the Jackson and Bernstein–Walsh theorems to the fractal setting, with piecewise polynomial and trigonometric approximation of degree n at level m (cells of size smaller than r^m, $0 < r < 1$):

Theorem (convergence rates)

f analytic: $\|f - P\|_2 = \mathcal{O}(r^{mn})$, $f \in C^k$: $\|f - P\|_2 = \mathcal{O}(n^{-k} r^{km})$.

The error decays exponentially for analytic targets, and polynomially in the degree while geometrically in the level for C^k targets.

General Measure Spaces & Non-Constant Generators

The classical density of simple functions extends to any metric space (X, d) with a finite Radon measure μ. By leveraging the uniform continuity of functions in $C_c(X)$ on their support, a fractal structure $\mathbf{\Gamma}$ with $\operatorname{diam}(\Gamma_n) \to 0$ ensures that $\sum_{n\in\mathbb{N}} V_n = \operatorname{span}\{\mathbf{1}_A : A \in \Gamma_n\}$ is dense in $L^2(X)$, removing any compactness requirement for the underlying space.

The single-generator rigidity observed in the interval case breaks down under asymmetric rescalings $\sigma_A : A \to X$. If the rescaling mappings compress mass across consecutive levels, non-constant functions can achieve universal density. The next example shows this type of phenomenon.

Example

On $X = [0,1]$ with Lebesgue measure, let $u = \mathbf{1}_{[0,1/2]}$. Constructing asymmetric, piecewise linear rescalings σ_A that compress mass into the first half of each cell yields a local L^2-error of $\frac{|A|}{n+1}$ when approximating $\mathbf{1}_X$. Summing over level n gives a total error smaller than $\frac{\mu(X)}{n+1} \to 0$, proving density.

Thus, to restore a wavelet-like framework, future lines of work must either enforce geometric measure-distortion bounds such as $\int_X f\, d(\mu \circ \sigma_A) \ge c\mu(A) \int_X f\, d\mu$, or deeply analyse which conditions the maps $\sigma_A : A \to X$ must satisfy to preserve these results.

Conclusions

Fractal structures provide a single language for approximation and decomposition in L^2: Wavelet constructions with compact domains (such as Haar wavelet) appear as special cases, sharp rigidity and existence theorems delimit which generating families work, and the convergence is fully quantified with polynomial and trigonometric families.

Conflict of Interest

The authors declare no conflicts of interest.

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