IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

MDPI

t-g-Radical Supplemented Modules

Hamdullah ÖZKAYA
Department of Mathematics
Bursa Technical University
Ayten PEKİN
Department of Mathematics
Istanbul University

INTRODUCTION & AIM

Motivation

Classical supplemented modules require every $N \leq M$ to have a supplement. g-radical supplements weaken intersection to $N \cap J \leq Rad\_g(J)$. t-sum terms generalize direct summands. This work unifies both.

Core Definitions

t-Sum Term (Def.)

$J \leq M$ is a t-sum term of M if $\exists\, K \leq M$ such that:

$\mathbf{J + K = M, \quad J \cap K \ll J, \quad J \cap K \ll K}$

Every direct summand is a t-sum term; t-sum terms allow small (<<) intersections.

SSP Property (Def.)

M has SSP (Summand Sum Property) if the sum of any two t-sum terms of M is again a t-sum term of M.

Definition 35 — t-g-Radical Supplemented Module

M is t-g-radical supplemented if every $N \leq M$ has a g-radical supplement that is also a t-sum term:

$\mathbf{\exists\, J \leq_t M: \ N + J = M \ \text{ and } \ N \cap J \leq Rad\_g(J)}$

Background (Zöschinger)

Thm 21: $Rad\_g(K) = K \cap Rad\_g(M)$ for every t-sum term K of M.

Thm 22: $K \ll M \iff K \ll V$ for every t-sum term V with $K \leq V \leq M$.

Goals

① Closure under t-sums, quotients, homomorphic images.
② Classify standard module classes (simple, local, semisimple, hollow).
③ Show the class strictly extends supplemented modules.
④ Determine when the SSP property preserves the class under quotients.

METHOD

Proof Strategies

Theorem 23 — Implies Supplemented

$Rad\_g(J) = J \cap Rad\_g(M) \leq Rad\_g(M) \ll M \implies H \cap J \ll J \implies$ M supplemented.

Theorem 24 — Closure Under t-Sums

$M = M_1 +_t M_2$. For $W \leq M$, find $S \leq_t M_2$ with $(M_1+W) \cap M_2+S=M_2$. Then $W+M_1+S=M$.

For $T \leq_t M_1$ with $M_1 \cap (W+S)+T=M_1$: $W+T+S=M$ and:

$\mathbf{W \cap (T+S) \leq Rad\_g(T)+Rad\_g(S) \leq Rad\_g(T+S)}$ ✓

T+S is a t-sum term by [Kosar 2014, Prop. 4.1.7].

Theorem 25 — Quotient Modules

If (T+H)/H is a t-sum term of M/H for every t-sum term T of M:

(S+H)/H is a g-radical supplement of W/H in M/H

By [Kosar 2019, Lemma 8], $H \leq W$ case.

Theorem 26 — Distributive Modules

Distributivity gives: $(P+H)/H \cap (S+H)/H = (P \cap S+H)/H$.

$P \cap S \ll P$ and $P \cap S \ll S \implies (P \cap S+H)/H \ll (P+H)/H$ and $\ll (S+H)/H$.

So (P+H)/H and (S+H)/H form a t-sum in M/H ⟹ Theorem 25 applies.

Theorem 27 & Corollary 9 — t-Sum Term Quotients

If H is a t-sum term and M_1,M_2 t-sum $\implies H = (H \cap M_1)+\_t\,(H \cap M_2)$:

$(M_1+H)/H$ and $(M_2+H)/H$ form a t-sum in M/H, so Theorem 25 applies.

Corollary 9: For epimorphism $f:M \to N$, if ker(f) respects t-sums, then N is t-g-rad. supp.

Theorem 28 — SSP Property

SSP + t-g-rad. supplemented: Q is g-rad. supplement, Q+H is t-sum term (by SSP),

(Q+H)/H is a g-radical supplement of W/H in M/H ✓

RESULTS & DISCUSSION

Main Theorems

Thm 23 — Implies Supplemented

$Rad\_g(M) \ll M$ + t-g-rad. supp. $\implies$ M supplemented.

Cor. 6: M f.g. + t-g-rad. supp. ⟹ M t-g-supplemented.

Thm 24 — Closure Under t-Sums

$\mathbf{M_1 +_t M_2}$**, each t-g-rad. supp.** $\implies$ **M t-g-rad. supp.**

Cor. 7: Finite t-sums $M = M_1 +_t \cdots +_t M_n$ inherit the property.

Thm 25 — Quotient Modules

$(T+H)/H \leq_t M/H$ for all t-sum terms T $\implies$ M/H t-g-rad. supp.

Thm 26 — Distributive + Cor. 8 — Homomorphic Images

M distributive + t-g-rad. supp. $\implies$ every M/H t-g-rad. supp.

Cor. 8: Every homomorphic image is t-g-radical supplemented.

Thm 27 — t-Sum Term Quotients + Cor. 9

M_1,M_2 t-sum, H t-sum with $H=(H \cap M_1)+\_t(H \cap M_2) \implies$ M/H t-g-rad. supp.

Cor. 9: Epimorphism $f:M \to N$ where ker(f) respects t-sums ⟹ N t-g-rad. supp.

Thm 28 — SSP Property

M has SSP + t-g-rad. supp. $\implies$ M/H t-g-rad. supp. for every t-sum term H.

Lemma — t-Sum Terms Inherit the Property

Intersections of t-sum terms are t-sum terms $\implies$ every t-sum term $P \leq M$ is t-g-rad. supp.

Proof: $P = W+(P \cap S)$ by modular law; $W \cap (P \cap S) \leq Rad\_g(P \cap S)$ by Thm 21.

Strict Extension of Supplemented Modules

If $Rad\_g(M) \ll M$ the condition collapses to supplementedness. Otherwise:

ℚ: no maximal submodules ⟹ Rad_g(ℚ)=ℚ. Every $A \leq ℚ$ has K=ℚ as g-rad. supp.:

A ∩ ℚ = A ≤ ℚ = Rad_g(K) ✓ but ℚ is NOT supplemented

Examples and Non-Examples

Module Classes That Are t-g-Radical Supplemented

Hollow: every proper N, satisfies; $N \ll M$ then take J=M, $N \cap M \leq Rad(M) \leq Rad\_g(M)$. ✓

Local: unique max. submodule Q=Rad_g(M); H≠M ⟹ H ≤ Q; take J=M. ✓

Simple (ℤ/pℤ, ℤ/3ℤ, $\mathbb{F}_2$, ℤ[i]/(1+i)): trivial submodules ⟹ trivial supplements. ✓

Semisimple (ℤ/6ℤ ≅ ℤ/2ℤ⊕ℤ/3ℤ, ℤ/2ℤ⊕ℤ/2ℤ): every N is a direct summand = t-sum term. ✓

ℚ: Rad_g(ℚ)=ℚ; K=ℚ works for any N. ✓ (but NOT supplemented)

ℤ_{p^∞}: only K=ℤ_{p^∞} completes any C_m; $C_m \leq$ Rad_g(ℤ_{p^∞}). ✓

Semiperfect ring: f.g. $M = M_1 \oplus \cdots \oplus M_n$ with local $M_i \implies$ t-g-rad. supp. ✓

Dedekind D/I: D/I ≅ ⊕ D/𝔭ᵢ^eᵢ (local factors) ⟹ t-g-rad. supp. ✓

Classification Table

Module	Reason	t-g-rad. supp.
ℤ/pℤ	simple	✓
$\mathbb{Z}/p^n\mathbb{Z}$	local (hollow)	✓
ℤ/6ℤ, ℤ/12ℤ	semisimple / mixed	✓
ℤ/2ℤ⊕ℤ/2ℤ	semisimple	✓
ℚ	Rad_g(ℚ)=ℚ (not supplemented)	✓
ℤ_{p^∞}	Prüfer group (not supplemented)	✓
Semiperfect / D/I	local decomposition	✓
ℤ	Rad_g(ℤ)=0, no supplement	✗

CONCLUSION

① Closed under finite t-sums, quotients (with conditions), and homomorphic images.
② Simple, local, hollow, semisimple, and Prüfer group modules are all t-g-radical supplemented.
③ The class strictly extends supplemented modules: ℚ and ℤ_{p^∞} lie here but are NOT supplemented.
④ SSP property provides an additional sufficient condition for quotients to be t-g-radical supplemented.

FUTURE WORK / REFERENCES

Future Work

Extend to non-commutative rings; classify over Artinian, hereditary, and perfect ring classes; study interaction with the SSP property over specific module categories.

References

Zöschinger (1974) | Nebiyev & Kosar (2015) | Kosar (2019, Lem. 8) | Kosar (2014, Prop. 4.1.7) | Clark et al.