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Time-Embedded Information Geometry: A Compatible Extension of the Fisher–Rao Framework

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ABSTRACT

Classical information geometry provides a differential-geometric framework for statistical models using structures such as manifolds and the Fisher–Rao metric. However, time is typically treated as an external parameter, limiting the representation of evolving statistical systems.

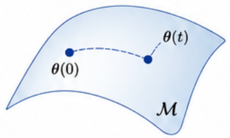
In this paper, we propose **Time-Embedded Information Geometry (TEIG)**, where temporal evolution is incorporated into the coordinate structure of statistical manifolds.

The proposed formulation preserves compatibility with classical Fisher–Rao geometry while enabling a geometric representation of time-dependent statistical configurations.

1. MOTIVATION

Classical Information Geometry

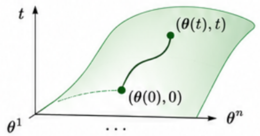
- Statistical model $\mathcal{M} = \{p(x|\theta)\}, \theta \in \Theta \subset \mathbb{R}^n$
- Manifold $\mathcal{M}$ with Fisher–Rao metric
- Time t is treated as an external parameter



Limitation
Temporal evolution is not represented within the geometric structure.

Time-Embedded Information Geometry (TEIG)

- Time is embedded into the coordinate system
- Evolving statistical systems are represented geometrically
- Compatible with Fisher–Rao framework



Advantage
Time-dependent statistical configurations are represented as geometric trajectories.

2. PRELIMINARIES

Let $\{p(x|\theta)\}$ be a regular parametric family with $\theta \in \Theta \subset \mathbb{R}^n$.

- Fisher–Rao metric

$$g_{ij}(\theta) = \mathrm{E}_\theta\left[\frac{\partial \ln p(x|\theta)}{\partial \theta^i}\frac{\partial \ln p(x|\theta)}{\partial \theta^j}\right]$$

- Levi-Civita connection

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial \theta^i} + \frac{\partial g_{il}}{\partial \theta^j} - \frac{\partial g_{ij}}{\partial \theta^l}\right)$$

- Geodesic equation

$$\frac{d^2\theta^k}{ds^2} + \Gamma^k_{ij}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds} = 0$$

These structures form the basis of classical information geometry.

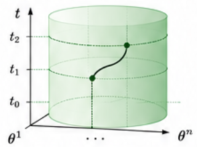
3. TIME-EMBEDDED INFORMATION GEOMETRY (TEIG)

We extend the parameter space by embedding time as an additional coordinate.

3.1 Extended Manifold

$\overline{\mathcal{M}} = \Theta \times \mathbb{R}_t$ with coordinates $(\theta^1, \dots, \theta^n, t)$

A point $(\theta, t) \in \overline{\mathcal{M}}$ represents the statistical state at time t.



3.2 Time-Embedded Metric

We define the metric on $\overline{\mathcal{M}}$ as

$$\bar{g} = \begin{pmatrix} g_{ij}(\theta,t) & g_{it}(\theta,t) \\ g_{tj}(\theta,t) & g_{tt}(\theta,t) \end{pmatrix}$$

- ☑ g_{ij} : Fisher–Rao metric on each time slice $t = \text{const.}$
- ☑ g_{it}, g_{jt} : capture temporal variation and coupling
- ☑ g_{tt} : positive-definiteness ensures a valid Riemannian metric.

3.3 Compatibility

If $g_{it} = 0$ and $g_{tt} = 1$, the metric reduces to the product manifold $\bar{g} = g \oplus 1$, preserving the classical Fisher–Rao geometry on each slice.

4. GEOMETRIC DYNAMICS

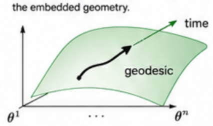
The evolution of a statistical system is represented by geodesics on $\overline{\mathcal{M}}$.

- Geodesic equation on $\overline{\mathcal{M}}$

$$\frac{d^2x^a}{ds^2} + \bar{\Gamma}^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0, \quad x^a = (\theta^1, \dots, \theta^n, t)$$

- Interpretation

The trajectory $(\theta(s), t(s))$ describes the most natural evolution of the statistical state under the embedded geometry.



5. EXAMPLE (GAUSSIAN FAMILY)

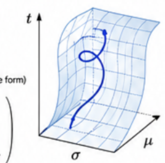
Consider $\mathcal{N}(\mu, \sigma^2)$ with parameters (μ, σ).

- Fisher–Rao metric (2D)

$$g = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}$$

- Time-embedded metric (example form)

$$\bar{g} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 & \alpha \\ 0 & \frac{2}{\sigma^2} & \beta \\ \alpha & \beta & 1+\gamma \end{pmatrix}$$



where α, β, γ model time-coupling.

Insight
Temporal changes in mean and variance appear as geometric curves on $\overline{\mathcal{M}}$.

6. PROPERTIES & BENEFITS

Compatibility
Includes classical Fisher–Rao geometry as a special case.

Intrinsic Time Representation
Time evolution is represented within the manifold.

Geometric Insight
Enables analysis of time-dependent statistical systems using differential geometry.

Applicability
Useful for non-stationary processes, adaptive systems, and control problems with uncertainty.

7. CONCLUSIONS

- We proposed Time-Embedded Information Geometry (TEIG), a natural extension of the Fisher–Rao framework that incorporates time into the coordinate structure of statistical manifolds.
- TEIG preserves compatibility with classical information geometry while enabling a geometric representation of time-dependent statistical configurations.
- This framework opens new possibilities for analyzing and controlling evolving statistical systems.

FUTURE WORK

- Characterization of temporal coupling terms (g_{it}, g_{tt})
- Relationship with thermodynamic and stochastic geometry
- Applications to control, estimation, and machine learning
- Numerical methods for geodesic computation on $\overline{\mathcal{M}}$