Fundamental Incompatibility of the Yang-Mills Mass Gap
and Asymptotic Freedom within Continuum Quantum Field Theory

Miltiadis Karazoupis

Ministry of Education and Religious Affairs, Greece | miltos.karazoupis@gmail.com | ORCID: 0009-0009-1778-6728

INTRODUCTION & AIM

The Millennium Prize Challenge: The Clay Mathematics Institute requires a rigorous proof of existence and a mass gap ($\Delta > 0$) for non-abelian Yang-Mills theory within the continuum framework of Quantum Field Theory (QFT).

The Theoretical Crisis: Standard formulations demand that the theory satisfies the Osterwalder-Schrader (OS) axioms while simultaneously ex-hibiting two fundamental physical phenomena:

- **The Mass Gap ($\Delta > 0$):** Essential for quark confinement.
- **Asymptotic Freedom:** The empirically verified logarithmic weakening of the strong force at high energies (UV regime).

Aim: To demonstrate that these three established requirements are mathematically incompatible when imposed simultaneously on a Yang-Mills theory formulated on a fundamental continuum spacetime R^4.

THE THREE PILLARS OF CONFLICT

This incompatibility arises by forcing the analytic structures of axiomatic QFT to simultaneously accommodate opposing infrared (IR) and ultraviolet (UV) behaviors:

- **1. OS Axioms & Reflection Positivity:** Ensures a well-defined relativistic QFT, mandating the Källén-Lehmann spectral representation for two-point functions.
- **2. The Mass Gap ($\Delta > 0$):** Physically restricts the support of the spectral density, forcing it to vanish for states below Δ^2.
- **3. Asymptotic Freedom:** Dictates specific high-momentum scaling laws derived from the Renormalization Group (RG) and Operator Product Expansion (OPE), requiring the coupling to vanish logarithmically.

METHODOLOGY: The Analytic Confrontation

We confront the spectral constraints of the mass gap against the RG asymptotics.

1. The Spectral Representation:
Consider the gauge-invariant, local operator $O(x) =: \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})(x):$, which has dimension $d = 4$. Reflection positivity and the mass gap ($\Delta > 0$) yield the spectral representation:

$$\tilde{S_2}(p^2) = P(p^2) + \int_{\Delta^2}^{\infty} dm^2 \frac{\rho_c(m^2)}{p^2 + m^2}$$

where $\rho_c(m^2) \geq 0$ is the spectral density and p^2 is the Euclidean momentum.

2. Constraint on Subtraction Polynomial:
Renormalization theory limits the maximum degree of the subtraction polynomial $P(p^2)$ based on the superficial degree of divergence (logarithmic for $d = 4$). $P(p^2)$ can be at most linear:

$$P(p^2) = a_0 + a_1 p^2$$

3. Spectral Asymptotic Behavior:
Expanding the integral for large p^2 (assuming convergence of the first moments of ρ_c for temperedness), the integral term $I(p^2) = O(1/p^2)$. Thus, the OS axioms dictate:

$$\tilde{S}_2(p^2) \sim a_1 p^2 + a_0 + O(1/p^2) \quad \text{as } p^2 \to \infty$$

RESULTS: Proof of Mathematical Contradiction

Asymptotic Freedom vs. Spectral Bounds

Asymptotic freedom and the Renormalization Group dictate the true UV asymptotic behavior of $\tilde{S}_2(p^2)$. For $O =: \mathrm{Tr}(F^2):$ RG predicts:

$$\tilde{S}_2(p^2) \sim C_0 \frac{p^2}{[\ln(p^2/\Lambda^2)]^k} \quad \text{as } p^2 \to \infty$$

where $C_0 \neq 0$ and $k > 0$.

Deriving the Contradiction

We compare the asymptotic behavior derived from the spectral representation (OS axioms) with that required by asymptotic freedom:

$$a_1 p^2 \quad \text{vs} \quad C_0 \frac{p^2}{[\ln(p^2/\Lambda^2)]^k}$$

- **The Match:** Since $1/[\ln(p^2/\Lambda^2)]^k \to 0$ as $p^2 \to \infty$, asymptotic equivalence strictly requires the linear coefficient a_1 to be zero ($a_1 = 0$).
- **The Paradox:** With $a_1 = 0$, the spectral behavior simplifies to $\tilde{S}_2(p^2) \sim a_0 + O(1/p^2)$. The function approaches a constant (or decays).
- **The Contradiction:** A function approaching a constant (a_0) cannot be asymptotically equivalent to the unbounded growth mandated by asymptotic freedom ($\sim p^2/[\ln(p^2/\Lambda^2)]^k$).

Conclusion: A continuum Yang-Mills theory on R^4 cannot simultaneously satisfy the OS axioms, possess a mass gap, and exhibit asymptotic freedom.

DISCUSSION: Implications for Physics & Spacetime

1. The Millennium Problem is Ill-Posed:
The central finding suggests that the Clay Mathematics Institute's challenge may not have a solution within its current formulation. The difficulty lies not in a lack of mathematical tools, but in the inherent inconsistency of the continuum QFT framework when forced to accommodate both IR and UV physical realities.

2. The Limits of Lattice Gauge Theory (LGT):
While LGT successfully simulates both the mass gap and asymptotic freedom at finite lattice spacing a, this proof indicates that the strict continuum limit ($a \to 0$) cannot satisfy the OS axioms while retaining both features.

3. The End of Continuum Spacetime:
The rigidity of analytic structures in continuum QFT causes this conflict. Resolving it necessitates abandoning the fundamental continuum assumption. A transition to discrete or quantum geometry is required to naturally reconcile gapped IR behavior with UV asymptotic freedom.

PREPRINT ACCESS & AUTHOR INFO

Read the Full Paper:

Scan the QR code to access the complete manuscript detailing the correlation function analysis and the fundamental limits of continuum Quantum Field Theory.

Contact:

Miltiadis Karazoupis
miltos.karazoupis@gmail.com
DOI: 10.20944/preprints202504.1268.v1