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Bridging Analytical and Numerical Methods for Fractional 4D Chaotic Systems.

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INTRODUCTION & AIM

Fractional calculus has emerged as a powerful mathematical framework for modeling real-world systems characterized by memory, nonlocality, and complex dynamical behaviors. Unlike classical integer-order models, fractional-order formulations provide a more flexible description of processes with long-term memory and hereditary effects, which are commonly observed in physics, engineering, biology, and information sciences. In particular, fractional-order chaotic and hyperchaotic systems have received considerable attention for their ability to capture richer dynamical features, making them highly relevant in areas such as secure communications, signal encryption, random number generation, and nonlinear control. In this work, we explore a four-dimensional fractional-order chaotic model by integrating the Residual Power Series Method (RPSM) and the Caputo fractional derivative (CFD). The CFD is employed to obtain highly accurate numerical simulations that reveal the sensitivity of chaotic attractors to changes in the fractional order, while the RPSM provides analytical approximations that are computationally efficient, stable, and capable of handling diverse initial conditions. This hybrid framework bridges the gap between purely numerical and purely analytical methods, ensuring both accuracy and mathematical insight. Our findings show that fractional order is a key parameter for tuning system behavior between chaotic and stable regimes. This study contributes new insights into the modeling of hyperchaotic systems, with potential relevance for engineering, cryptography, and nonlinear sciences.

METHOD

The fractional hyperchaotic system is governed by the following set of differential equations:

$$
\begin{aligned}
{}^{\mathrm{LC}}D_t^{\alpha} x_1 &= \rho_1 (y_1 - x_1),\\
{}^{\mathrm{LC}}D_t^{\alpha} y_1 &= \rho_2 x_1 - x_1 z_1 + \rho_3 y_1 - w_1,\\
{}^{\mathrm{LC}}D_t^{\alpha} z_1 &= x_1 y_1 - \rho_4 z_1,\\
{}^{\mathrm{LC}}D_t^{\alpha} w_1 &= x_1 + \rho_5,
\end{aligned}
$$

where x_1, y_1, z_1, and w_1 denote state variables, and ρ_1= 36, ρ_2 = −16, ρ_3 = 28, ρ_4 = 3, ρ_5 = 0.5 are parameters with real values.

The above fractional system has equilibria at constant states; hence, we set the Caputo-like derivatives to zero and solve the algebraic system :

$$
\begin{aligned}
\rho_1 (y_1 - x_1) &= 0,\\
\rho_2 x_1 - x_1 z_1 + \rho_3 y_1 - w_1 &= 0,\\
x_1 y_1 - \rho_4 z_1 &= 0,\\
x_1 + \rho_5 &= 0.
\end{aligned}
$$

Step 1. Immediate relations : $y_1 = x_1 = -\rho_5$.

Step 2. Solve for z_1 : $x_1^2 - \rho_4 z_1 = 0 \Rightarrow z_1 = \frac{x_1^2}{\rho_4}$.

Step 3. Solve for w_1 : Use $y_1 = x_1$ in the second equation and rearrange, we get $w_1 = x_1(\rho_2 + \rho_3 - z_1)$.

Step 4. Equilibrium point (parameters): Substituting $x_1 = -\rho_5$, we obtain

$$E_1 = (x_1^*, y_1^*, z_1^*, w_1^*) = \left(-\rho_5, -\rho_5, \frac{\rho_5^2}{\rho_4}, -\rho_5\left(\rho_2 + \rho_3 - \frac{\rho_5^2}{\rho_4}\right)\right).$$

Using numerical values with ρ_1 = 36, ρ_2 = −16, ρ_3 = 28, ρ_4 = 3, ρ_5 = 0.5, we get

$$E_1 = (x_1^*, y_1^*, z_1^*, w_1^*) = (-0.5, -0.5, 0.083333, -5.958333).$$

The corresponding Jacobian matrix of the system is given by

$$J = \begin{bmatrix} -\rho_1 & \rho_1 & 0 & 0 \\ \rho_2 - z_1 & \rho_3 & -x_1 & -1 \\ y_1 & x_1 & -\rho_4 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -36 & 36 & 0 & 0 \\ -16 - z_1 & 28 & -x_1 & -1 \\ y_1 & x_1 & -3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalues of the Jacobian at this equilibrium point are:

$$\lambda_1 \approx -25.1415, \quad \lambda_2 \approx 17.0184, \quad \lambda_3 \approx 0.0852, \quad \lambda_4 \approx -2.9622.$$

This result indicates that the equilibrium point E_1 behaves as a saddle point and, consequently, is unstable.

RESULTS & DISCUSSION

The system can be written in a compact form : ${}^{\mathrm{LC}}D_t^{\alpha(t)} x(t) = Ax(t) + F(x(t))$, where $x(t) = (x_1, y_1, z_1, w_1)^T$.

$$A = \begin{bmatrix} -36 & 36 & 0 & 0 \\ -16 & 28 & 0 & -1 \\ 0 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad F(x(t)) = \begin{bmatrix} 0 \\ -x_1 z_1 \\ x_1 y_1 \\ 0.5 \end{bmatrix}.$$

The linear part is governed by the constant matrix A, while F(x(t)) collects the nonlinear terms. Hence, the stability of the 4D hyperchaotic Chen system under variable-order derivatives can be guaranteed using the same framework established for the general fractional variable-order system. In particular, with the imposition of spectral conditions on the linear part of the system matrix and suitable bounds on the variable-order kernel, asymptotic stability of the system can be guaranteed.

The system involves four state variables: x1, y1, z1, and w1, and five real-valued parameters defined as :

$$\rho1 = 36, \rho2 = -16, \rho3 = 28, \rho4 = 3, \rho5 = 0.5.$$

The initial conditions for all cases are given by :

$$x1(0) = 0, y1(0) = 0, z1(0) = 8, w1(0) = 6$$

x1(0) = 0, y1(0) = 0, z1(0) = 8, w1(0) = 6.

The system is analyzed under three different scenarios of the variable-order function α(t) as follows:

- Case 1: $\alpha(t) = \tanh(t+1)$.
- Case 2: $\alpha(t) = 0.97 + 0.03\cos\left(\frac{t}{10}\right)$.
- Case 3: $\alpha(t) = \frac{1}{1+\exp(-t)}$.

CONCLUSION

In this study, to explore the complex dynamics of higher-dimensional systems, we first constructed a 4D hyperchaotic Chen system via variable-order fractional calculus using the Liouville–Caputo derivative. By using derivative applications of variable order, we gained a deeper understanding of the time dynamics and adaptability of fractional order systems. The findings suggest that system parameters and the order function play a crucial role in controlling hyperchaotic behavior and determining phase-space trajectories. Such findings provide a solid foundation for studying complex nonlinear dynamics in higher-order systems.

FUTURE WORK

In the future, we intend to extend this approach to solve new fractional models and compare it with other analytical and numerical approaches. This research will further advance the science and engineering practice of variable-order fractional systems.

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