Numerical Solution of Eighth-Order Boundary Value Problem Using Shifted Horadam Collocation Method

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Aim and Objectives

Aim: To employ a shifted Horadam collocation method to obtain numerical solutions of the eighth-order boundary value problem $[x \in (0,1)]$

$$v^{(8)}(x) = f(x, v(x), v'(x), v''(x), \dots, v^{(7)}(x))$$
$$v(0) = \xi_1,\ v(1) = \eta_1,\ v'(0) = \xi_2,\ v'(1) = \eta_2,$$
$$v''(0) = \xi_3,\ v''(1) = \eta_3,\ v'''(0) = \xi_4,\ v'''(1) = \eta_4.$$

Here ξ_i, η_i $(i = 1,2,3,4)$ are real constants.
Specific objectives: To

1. express solutions in shifted Horadam polynomial series;
2. collocate the resulting problem at roots of Horadam polynomials;
3. transform the collocation scheme into a set of algebraic equations;
4. compare numerical results with exact solutions and other published results.

Introduction

Boundary value problems (BVPs) appear in several applied sciences and other real-life phenomena such as fluid dynamics, astrophysics, hydrodynamics, electromagnetism, beam theory, astronomy, induction motors, solid mechanics, mathematical physics, astronomy, mathematical biology, economics, finance, and engineering. Specifically, eight-order BVPs appear in the deflection of beam theory and analysis, plate deflection theory, viscoelastic and inelastic flows, deformation of beam, and the physics of various hydrodynamic stability theory. Several authors have used different numerical techniques to solve eight-order BVPs such as [1] (reproducing kernel), [2] (Vieta-Lucas polynomials), [3] (Chebyshev polynomials). Other approaches include wavelets techniques. To the best of the author's knowledge, Horadam collocation method has not been applied to the proposed problem.

Results & Discussion

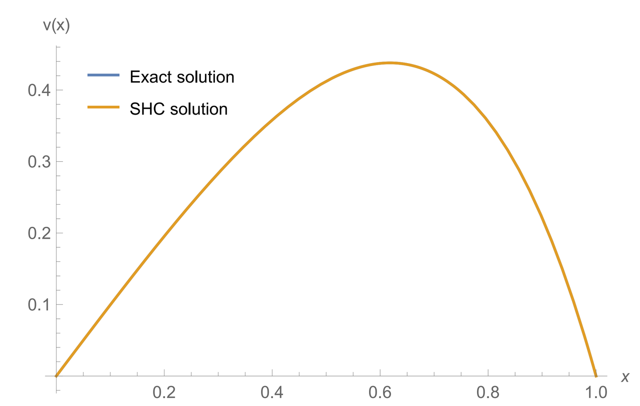

Figure 1: Exact & appr. solutions for Example 1

x	Present Meth.	Meth. [2]	Meth. [3]
0.1	4.06×10^{-17}	5.55×10^{-9}	1.02×10^{-6}
0.3	5.37×10^{-17}	4.55×10^{-8}	1.46×10^{-6}
0.5	5.90×10^{-17}	1.60×10^{-7}	6.12×10^{-9}
0.7	3.77×10^{-17}	2.11×10^{-7}	1.44×10^{-6}
0.9	1.48×10^{-16}	1.18×10^{-8}	9.88×10^{-7}

Table 1: Absolute errors for Example 1

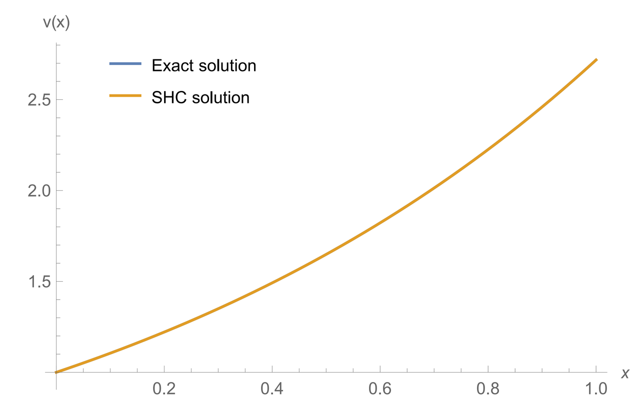

Figure 2: Exact & appr. solutions for Example 2

x	Present Meth.	Meth. [1]	Meth. [2]
0.2	1.03×10^{-17}	3.07×10^{-8}	5.75×10^{-10}
0.4	5.34×10^{-19}	4.97×10^{-8}	8.45×10^{-12}
0.6	3.55×10^{-17}	4.97×10^{-8}	3.05×10^{-9}
0.8	2.27×10^{-16}	3.08×10^{-8}	1.22×10^{-9}
0.9	1.94×10^{-16}	1.61×10^{-7}	1.55×10^{-10}

Table 2: Absolute errors for Example 2

The numerical solutions obtained using the present method outperform other existing results.

Methodology: SHCM

The mth-degree shifted Horadam polynomials are defined recursively by $(m = 2, 3, \dots)$

$$\mathsf{H}_m(x) = -2(2x-1)\mathsf{H}_{m-1}(x) - \mathsf{H}_{m-2}(x)$$
$$\mathsf{H}_0(x) = 1, \qquad \mathsf{H}_1(x) = -2(2x-1).$$

For $m = 0, 1, 2, \dots$, one computes $\mathsf{H}_m^{(r)}(0), \mathsf{H}_m^{(r)}(1)$. We have the assumed solution of the governing eight-order BVP:

$$v_N(x) = \sum_{m=0}^{N} c_m \mathsf{H}_m(x) \qquad (N \in \mathbb{N}).$$

The collocation points are the zeros of $\mathsf{H}_{N-7}(x)$.

Illustrative Examples

Example 1: Consider the linear BVP $(0 < x < 1)$

$$v^{(8)}(x) + xv(x) = -(48 + 15x + x^3)e^x$$
$$v(0) = 0,\ v(1) = 0,\ v'(0) = 1,\ v'(1) = -e,$$
$$v''(0) = 0,\ v''(1) = -4e,\ v'''(0) = -3,\ v'''(1) = -9e.$$

Exact solution: $v(x) = x(1-x)e^x$. We set $N = 16$.
Example 2: Given the nonlinear BVP $(0 < x < 1)$

$$v^{(8)}(x) = e^{-x}v^2(x) \qquad (0 < x < 1)$$
$$v(0) = 1,\ v(1) = e,\ v'(0) = 1,\ v'(1) = e,$$
$$v''(0) = 0,\ v''(1) = e,\ v'''(0) = 1,\ v'''(1) = e.$$

Exact solution: $v(x) = e^x$. We set $N = 16$.

Conclusion

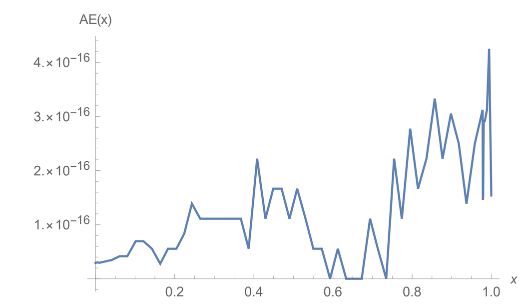

Figure 3: Absolute errors for Example 1

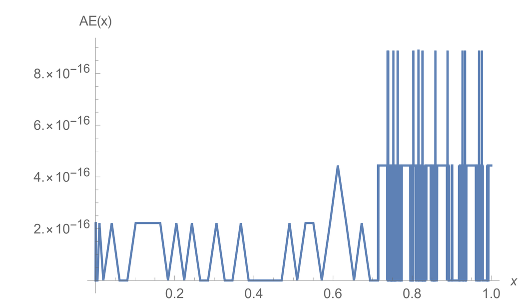

Figure 4: Absolute errors for Example 2

- This paper has applied a collocation technique based on shifted Horadam polynomials to obtain approximate solutions of eighth-order BVPs in ODEs.
- Horadam polynomials are a family of orthogonal polynomials of which Fibonacci polynomials, Pell polynomials, Lucas polynomials, Pell-Lucas polynomials, Chebyshev polynomials of the first kind, and Chebyshev polynomials of the second kind are special cases.
- More accurate results can be obtained as N increases.

References

[1] Akram, G., Rehman, H.U., *Numer. Algor.*, **62**, 527–540 (2013).
[2] Kumar, R., Aeri, S., Sharma, P., Lecture Notes in Networks Systems, Springer, 2022.
[3] Raji, M.T., Ishola, C.Y., Babalola, O., Ayoola, T., Momoh, N., *Math. Comput. Sci.*, **4**, 18–28 (2023).

Future Research

One can also apply collocation methods based on other members of Horadam polynomials to the present problems. Other families of orthogonal polynomials such as Jacobi polynomials (of which Chebyshev polynomials of the first, second, third, & fourth kinds and Legendre polynomials are special cases) can also be applied to the present problems and their fractional order versions. The present method can as well be used to solve initial and boundary value problems in partial differential equations arising in real-world phenomena.

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