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Scaling invariance for the diffusion coefficient in a billiard system

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INTRODUCTION & AIM

Diffusion is a fundamental stochastic process used to describe transport across diverse systems, from epidemiology to particle physics. This study investigates the diffusion coefficient D in a time-dependent oval billiard, a dynamical system characterized by a mixed phase space and the phenomenon of Fermi Acceleration. We aim to analyze the transition from unbounded energy growth in the conservative case to a stationary state induced by dissipation. By deriving analytical expressions for the probability distribution function and D, we demonstrate that the diffusion coefficient exhibits scaling invariance and can be described by a generalized homogeneous function. This approach allows us to define scaling laws and critical exponents, providing a novel framework for understanding the mechanisms underlying phase transitions in chaotic billiards.

METHOD

We studied a time-dependent oval-shaped billiard defined by the boundary $R_b(\theta,t) = 1+\epsilon[1+\eta\cos(t)]\cos(p\theta)$. *The parameter* ϵ *controls the geometry and non-integrability (chaos), while* η *dictates the amplitude of the temporal perturbation.*

The particle's dynamics are described by a four-dimensional nonlinear mapping $T(\theta_n, \alpha_n, |\overrightarrow{V_n}|, t_n)$. *To suppress the unlimited energy growth (Fermi Acceleration) and induce a phase transition, dissipation was introduced via a restitution coefficient* $\gamma < 1$, *which affects the normal component of the particle's velocity during collisions. Assuming a fully chaotic phase space, the stochastic spread of particles was modeled using the diffusion to obtain the probability distribution function P(V,n). This allowed us to analytically derive the mean squared displacement and formulate the diffusion coefficient D(n) as a function of the collision number and control parameters.*

$$\frac{\partial P(V,n)}{\partial n} = \frac{\partial D}{\partial V}\frac{\partial P}{\partial V} + D\frac{\partial^2 D}{\partial V^2}.$$

$$P(V,n) = \frac{\tau}{\sqrt{4\pi D n}}\left[e^{\left(\frac{-(V-V_0)^2}{4Dn}\right)} - e^{\left(\frac{-(V+V_0)^2}{4Dn}\right)}\right]$$

RESULTS & DISCUSSION

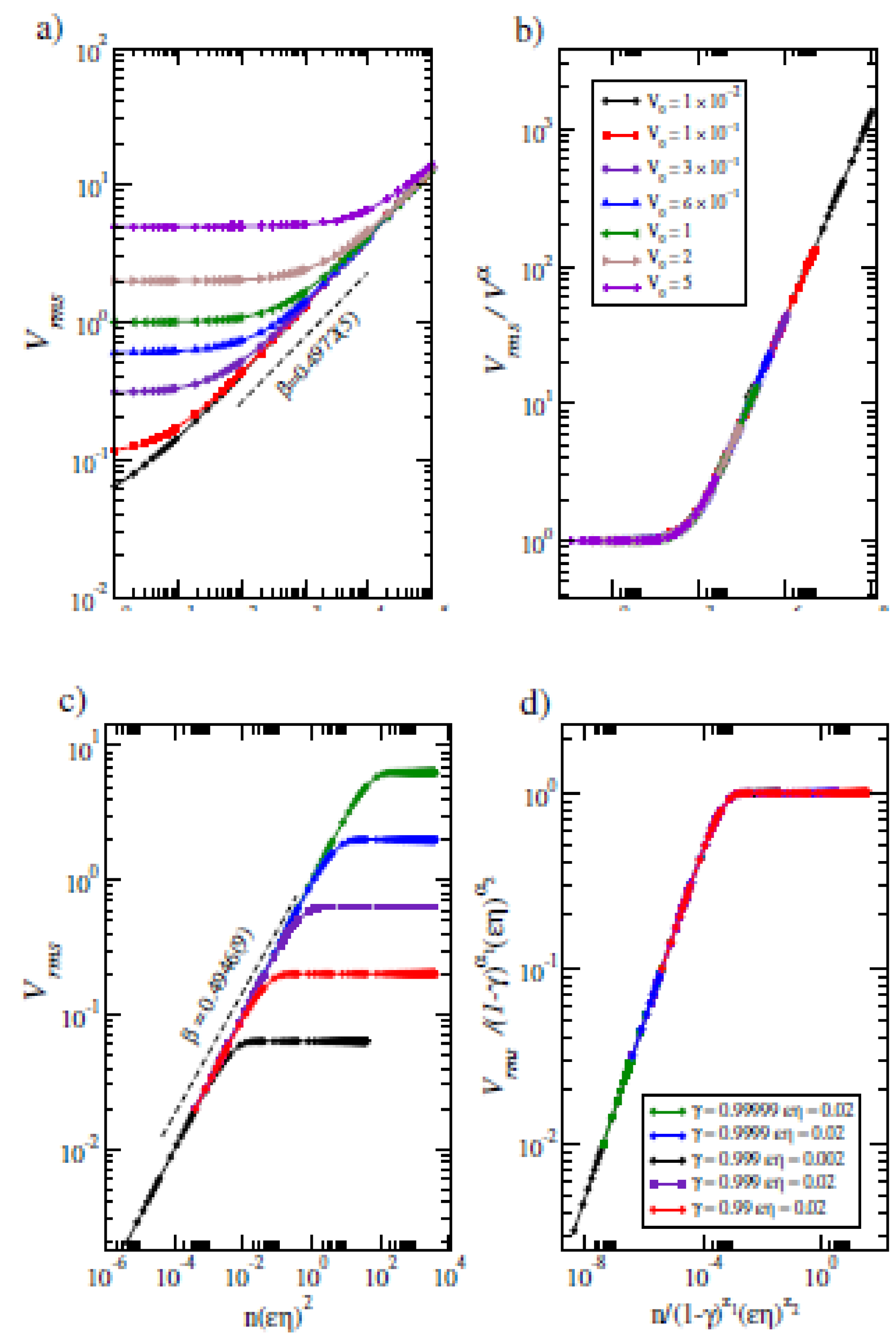

FIG. 1. (Color online) V_{rms} vs. n for (a) the conservative case with $\epsilon = 0.08$, $p = 3$, $\eta = 0.5$ and different values of V_0 and (c) the dissipative case for initial velocity $V_0 = 10^{-5}$ for different values of γ and $\eta\epsilon$. (b) and (d) present the overlap of the curves in (a) and (c), respectively, onto single universal curves using the adequate scaling transformations.

$$D = \frac{1}{n+1}\left(\overline{V_0^2} - \frac{(1+\gamma)}{4(1-\gamma)}\eta^2\epsilon^2\right) \times \left[\frac{1-e^{\frac{(\gamma^2-1)(n+1)}{2}}}{1-e^{\frac{(\gamma^2-1)}{2}}}\right]\frac{(\gamma^2-1)}{4}.$$

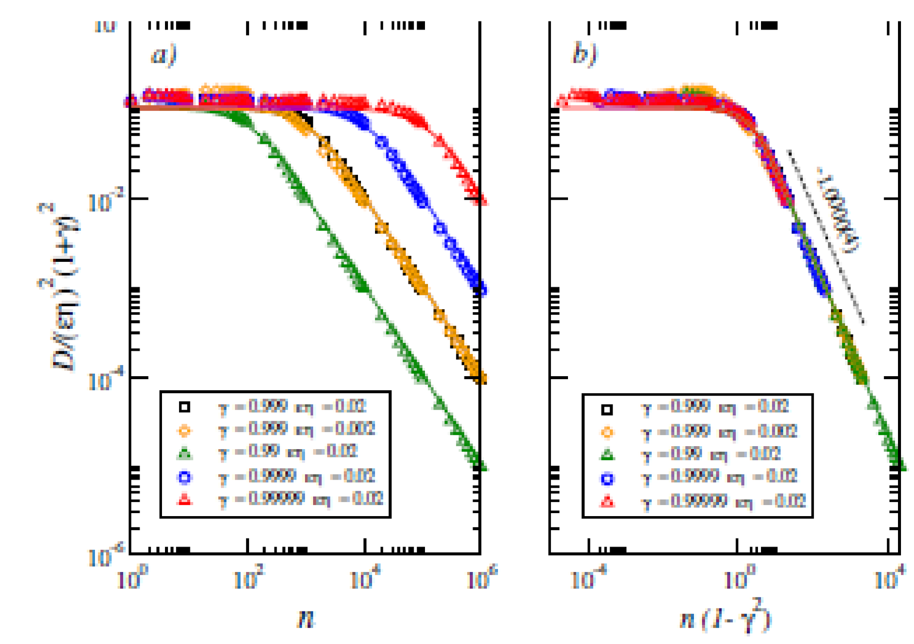

FIG. 2. Plot of curves of $D/(\epsilon\eta)^2(1+\gamma)^2$ (a) vs. n for different control parameters and (b) vs. $n(1-\gamma^2)$ overlapping the curves in (a) onto a single universal plot. In both cases, the symbols represent numerical curves while the continuous lines are obtained from equation (17).

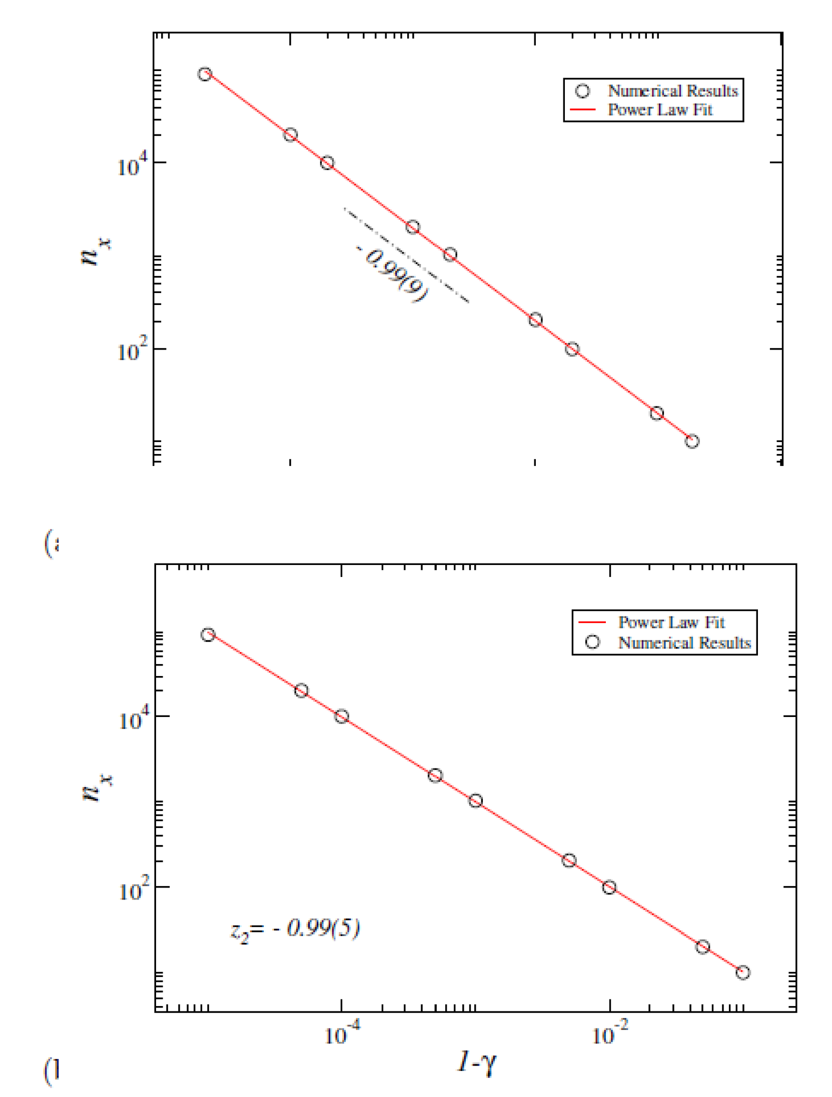

FIG. 3. (a) Plot of $D(n)$ vs. $(1-\gamma^2)$. A power law fitting yielded an exponent of $-0.99(9)$. (b) Plot of $D(n)$ vs. $(1-\gamma)$ with a power law fit giving $z_2 = -0.99(5)$.

CONCLUSION

1. *Dissipation suppresses Fermi acceleration, transitioning the system to a stationary state.*
2. *The diffusion coefficient D(n) is constant initially but decays after a crossover iteration. Analytical solutions for D(n) match the numerical data perfectly.*
3. *D(n) exhibits strong scaling invariance concerning the system's control parameters.*
4. *A generalized homogeneous function accurately describes this scale-invariant behavior.*
5. *Analytical critical exponents, including* $\beta = -1$, *confirm the phase transition mechanics.*

FUTURE WORK / REFERENCES

This work is part of the project "Properties of phase transitions in stochastic billiards" , aiming to extend the current methodology to other systems. The project, was awarded a Fulbright Doctoral Dissertation Research Award/2026. Authors declare funding from CAPES and FAPESP

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