Tsallis Nonextensive Entropy and Holographic Dark Energy: A Thermodynamic Approach to $f(G)$ Gravity Reconstruction

Bhagyashree Chandra[1], **Surajit Chattopadhyay**[2], **I.G. Salako**[3]

[1,3]Amity University Kolkata, India [2]Institute of Mathematics and Physical Sciences, Republic of Benin

E-mail: bhagyashree.chandra@s.amity.edu

Introduction

Observations from Type Ia Supernovae, Cosmic Microwave Background radiation and Baryon Acoustic Oscillations indicate that the Universe is presently undergoing an accelerated expansion phase. The standard ΛCDM model explains this phenomenon through a cosmological constant; however, it faces several theoretical difficulties such as the fine-tuning problem, coincidence problem and vacuum energy discrepancy.

An alternative approach is provided by modified theories of gravity, where the late-time acceleration emerges naturally from the geometric sector of spacetime. Among these theories, Gauss–Bonnet gravity has attracted considerable attention because higher-order curvature corrections may become important during different stages of cosmic evolution.

Recently, the holographic principle has motivated various dark energy models. Tsallis Holographic Dark Energy (THDE), based on non-extensive entropy, extends the conventional holographic framework and provides richer cosmological dynamics. Motivated by these developments, we reconstruct the functional form of $f(G)$ gravity from THDE by adopting the Granda–Oliveros infrared cutoff and a hybrid scale factor capable of describing both decelerated and accelerated epochs.

The reconstructed model is then subjected to detailed cosmological and thermodynamic investigations including stability analysis, equation of state evolution and verification of the Generalized Second Law of Thermodynamics.

Objectives

- Construct a viable reconstructed $f(G)$ gravity model using Tsallis Holographic Dark Energy.
- Employ the Granda–Oliveros infrared cutoff to connect holographic dark energy with local cosmological quantities.
- Study the cosmological evolution generated by a hybrid scale factor that unifies early decelerated expansion and late accelerated expansion.
- Examine the behavior of the reconstructed function $f(G)$ and its derivatives.
- Investigate the effective equation of state parameter and determine the corresponding cosmological phase.
- Analyze dynamical stability through the squared sound speed and adiabatic index.
- Test thermodynamic consistency through the Generalized Second Law of Thermodynamics at the GO horizon.
- Explore the impact of the non-extensive parameter on the evolution of the reconstructed cosmological model.

Methodology

1. Adopt Tsallis HDE with GO infrared cutoff.
2. Choose hybrid scale factor:
$$a(t) = e^{\psi t} t^n$$
3. Compute dynamical and curvature quantities:
$$H,\ \dot{H},\ G,\ \dot{G}$$
4. Numerically solve the reconstruction equation for $f(G)$.
5. Obtain effective density, pressure and EoS parameter:
$$\rho_G,\ p_G,\ w_G$$
6. Analyze stability using Squared sound speed and Adiabatic Index:
$$v_s^2 = \frac{dp}{d\rho}, \qquad \Gamma = \frac{\rho + p}{p} v_s^2$$
7. Test GSL of thermodynamics at: GO Horizon

Theoretical Framework

For the flat FRW metric,
$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$
the Gauss–Bonnet invariant becomes
$$G = 24H^2(H^2 + \dot{H})$$
The modified Friedmann equations are
$$3H^2 = \kappa^2(\rho_m + \rho_G)$$
$$-2\dot{H} - 3H^2 = \kappa^2(p_m + p_G)$$
where effective energy density and pressure:
$$\rho_G = \frac{1}{2\kappa^2}\left[-f(G) + Gf_G - 24H^3\dot{f}_G\right]$$
$$p_G = \frac{1}{2\kappa^2}\left[f(G) - Gf_G + 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G\right]$$

Results and Discussion

Tsallis HDE density:
$$\rho_{THDE} \propto L^{2q-4}$$
, q being non-extensive parameter.

GO infrared cutoff:
$$L_{GO} = (\alpha H^2 + \beta \dot{H})^{-1/2}$$
Thermal equilibrium condition:
$$\rho_G = \rho_{THDE}$$
, leading to the reconstruction equation:
$$-f(G) + Gf_G - 24H^3 f_{GG}\dot{G} = 2\rho_{THDE}$$

Model Parameters Used

Parameter	Value
ψ	0.25
n	1.36
α	2.5
β	0.05
δ	1.1
B	0.05
γ	0.5
ρ_{m0}	0.1
q	0.02

These parameter values are adopted throughout the numerical reconstruction and cosmological analysis unless otherwise stated.

Evolution of Reconstructed $f(G)$

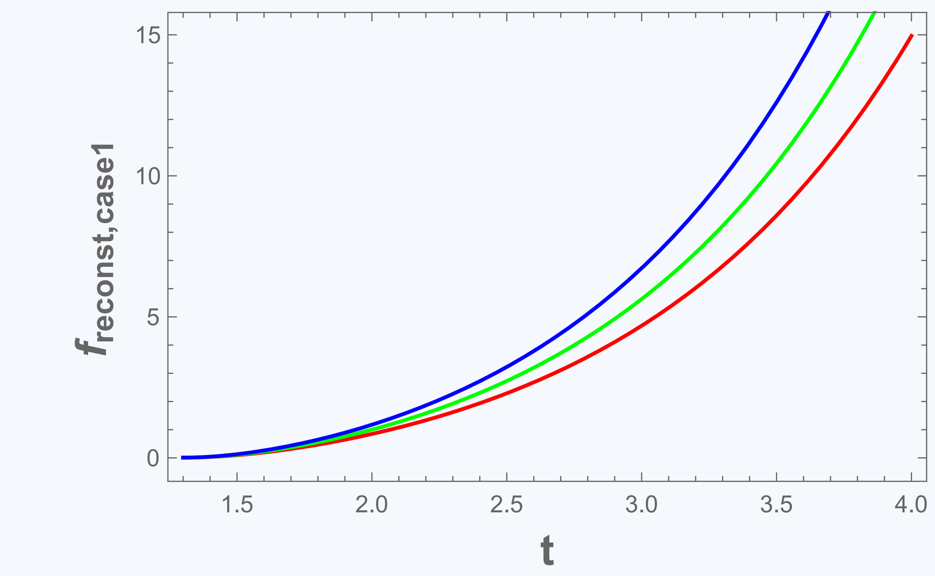


Figure 1: Reconstructed $f(G)$ versus cosmic time

- At early times:
$$f(G) \to 0$$
recovering standard General Relativity.
- At late times, modified gravity effects become significant and drive acceleration.
- The different trajectories corresponding to distinct parameter values reveal the sensitivity of the reconstruction to the model parameters while preserving the same qualitative cosmological behaviour.

Equation of State Parameter

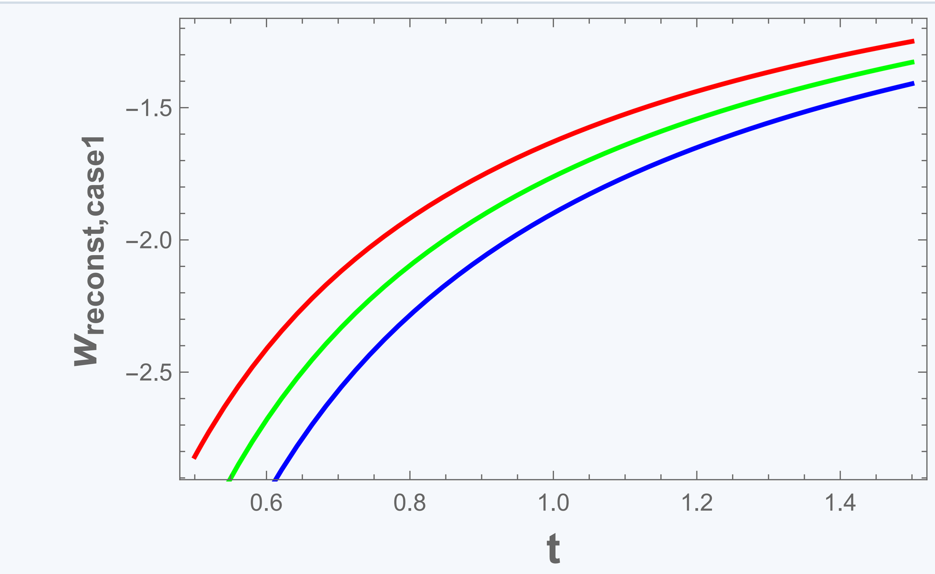


Figure 2: Evolution of reconstructed EoS parameter w_G

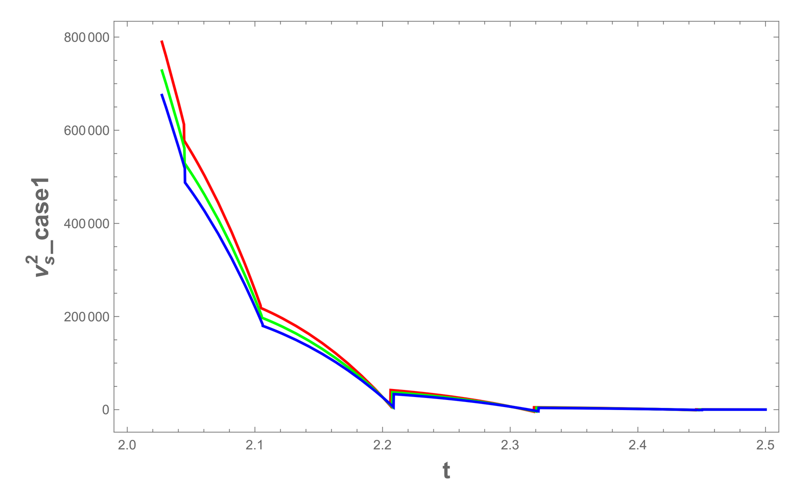
- Phantom regime:
$$w_G < -1$$
- Indicates super-accelerated expansion
- Stable accelerated phase achieved at late times

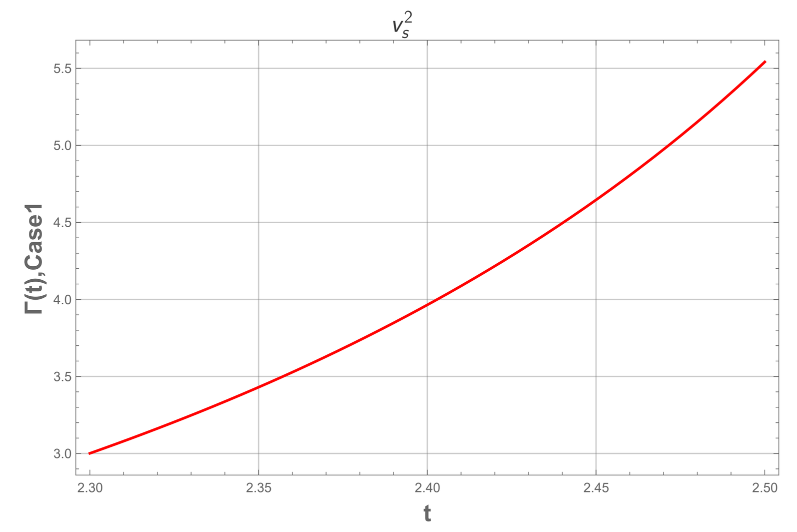
The effective equation of state parameter remains below the cosmological constant boundary throughout the evolution, indicating a phantom-like accelerating Universe.

The persistence of the phantom regime suggests that the reconstructed geometric dark energy possesses sufficient negative pressure to drive the observed acceleration. Moreover, the smooth evolution of w_G indicates the absence of sudden singular behaviour and supports the physical viability of the reconstructed model.

The parameter dependence of the trajectories demonstrates that the non-extensive features of Tsallis entropy significantly influence the quantitative evolution of the cosmic equation of state.

Dynamical Stability Analysis





The Squared sound speed and the Adiabatic index:
$$v_s^2 = \frac{dp}{d\rho}, \quad \Gamma = \frac{\rho + p}{p} v_s^2$$

- $v_s^2 > 0$ ensures classical stability.
- $\Gamma > \frac{4}{3}$ ensures stability against gravitational collapse.

The consistency between these two independent stability indicators provides strong evidence for the physical viability of the reconstructed $f(G)$ model.

Energy Conditions Analysis

Energy conditions provide important criteria for assessing the physical viability of cosmological models by constraining the effective energy density and pressure generated by the modified gravity sector.

For the reconstructed model, the following conditions are examined:
$$\text{NEC}: \quad \rho_G + p_G \geq 0,$$
$$\text{WEC}: \quad \rho_G \geq 0, \qquad \rho_G + p_G \geq 0,$$
$$\text{SEC}: \quad \rho_G + 3p_G \geq 0,$$
$$\text{DEC}: \quad \rho_G \geq |p_G|.$$
The analysis reveals that the reconstructed model satisfies the physically relevant energy constraints during the cosmic evolution while maintaining accelerated expansion. The observed behaviour is consistent with the phantom-like nature of the effective dark energy sector and supports the viability of the reconstructed $f(G)$ framework.

The fulfillment of these conditions, together with the positivity of the squared sound speed and the adiabatic index, provides additional evidence for the physical consistency of the model.

Thermodynamic Analysis

Generalized Second Law:
$$\dot{S}_{total} = \dot{S}_{horizon} + \dot{S}_{matter} \geq 0$$
Tsallis non-extensive entropy:
$$S_T = \gamma A^q$$
Matter entropy from Gibbs relation:
$$T_h dS_m = dE_m + p_G dV$$

GSL at GO Horizon

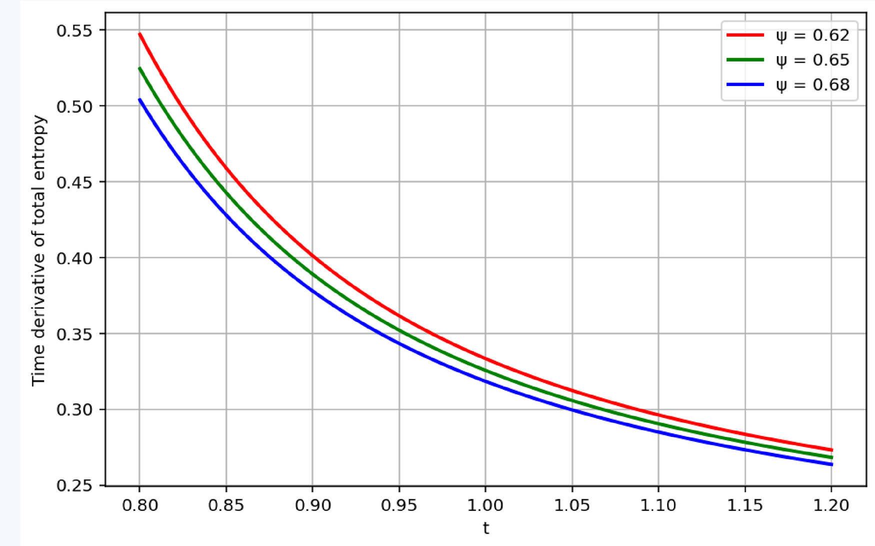


Figure 5: Evolution of $\dot{S}_{total}$

- Total entropy remains positive
- GSLT satisfied throughout evolution
- Thermodynamic consistency established

Physical Interpretation of the Results

The reconstructed $f(G)$ model successfully describes the observed accelerated expansion of the Universe through purely geometric effects. Unlike conventional dark energy models, the present framework attributes cosmic acceleration to higher-order curvature corrections encoded in the Gauss–Bonnet invariant.

The hybrid scale factor allows a smooth transition from an early matter-dominated epoch to a late-time accelerated phase. Consequently, the model reproduces the essential features of the observed Universe while avoiding the need for an explicit cosmological constant.

The reconstructed equation of state remains in the phantom regime, indicating the presence of sufficiently negative effective pressure to drive cosmic acceleration. The smooth behaviour of all cosmological quantities further suggests the absence of pathological instabilities during the evolution.

Moreover, the positivity of the squared sound speed and the fulfillment of the adiabatic stability criterion establish the dynamical robustness of the reconstructed model. The validity of the Generalized Second Law of Thermodynamics provides additional support for the thermodynamic consistency of the scenario.

Overall, the results indicate that Tsallis holographic dark energy can serve as a viable foundation for reconstructing stable and observationally consistent modified gravity models.

Conclusion: Key Findings and Relevance

- A viable reconstruction of $f(G)$ gravity has been achieved from Tsallis Holographic Dark Energy.
- The Granda–Oliveros cutoff successfully links holographic dark energy with local cosmological quantities.
- The hybrid scale factor provides a unified description of early and late-time cosmic evolution.
- The reconstructed function evolves smoothly and remains free from pathological behaviour.
- Geometric modifications become increasingly important at late times.
- The effective equation of state remains in the phantom regime.
- Accelerated expansion is generated purely from geometry.
- Positive squared sound speed confirms classical stability.
- The adiabatic index satisfies the gravitational stability criterion.
- The Generalized Second Law remains valid throughout the cosmic evolution.
- The model approaches thermodynamic equilibrium at late times.
- Overall, the reconstructed model represents a cosmologically viable, dynamically stable and thermodynamically consistent description of dark energy.

Future Scope

Future investigations may include observational constraints using Supernovae, BAO and CMB datasets to estimate the model parameters more precisely. The reconstructed framework can also be extended to anisotropic cosmologies, interacting dark energy scenarios and other modified gravity theories. In addition, the thermodynamic behaviour of the model may be explored using alternative horizon choices such as the apparent horizon and future event horizon to further examine the robustness of the Generalized Second Law.

Key References

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