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Analytical solution to geodesic equation in non-commutative gauge theory of gravity

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INTRODUCTION & AIM

Quantum gravity remains one of the most challenging and unresolved issues in modern physics. A promising approach to address this problem is through the introduction of NC geometry, which states that the quantization of spacetime naturally leads to the quanti-zation of gravity. This approach parallels quantum mechanics, where the commutation relations between observables define the rules of quantization. In the NC framework, spacetime coordinates follow non-trivial commutation relations given by $[x_\mu, x_\nu] = i\Theta_{\mu\nu}$, where $\Theta_{\mu\nu}$ is the NC parameter, encoding the fundamental discretization of spacetime. This relation introduces a modified Heisenberg uncertainty principle, expressed as:

$$\Delta x^\mu . \Delta x^\nu \geq \frac{1}{2}\Theta^{\mu\nu},$$

Our aim is to use NC gauge theory of gravity (GTG) [1] to construct a deformed Schwarzschild metric [2] and study how NC corrections affect the motion of a massive test particle. We derive the corrected effective potential and geodesic equation up to second order in the NC parameter, obtain an approximate analytical solution in the weak-field regime [3], and analyze the resulting orbital trajectories.

METHOD

In the NC spacetime, the coordinates satisfy the commutation relation:

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu},$$

where $\Theta_{\mu\nu}$ is an anti-symmetric real matrix which determines the fundamental cell discret-ization of space-time. In our work we use the NC formalism in the GTG to describe the background of our calculation [1]

$$e^a_{\mu\nu\rho} = \frac{1}{4}\left[\omega^{ac}_\nu \partial_\rho e^d_\mu + \left(\partial_\rho \omega^{ac}_\mu + F^{ac}_{\rho\mu}\right) e^d_\nu\right]\eta_{cd},$$

and

$$\begin{aligned} e^a_{\mu\nu\rho\lambda\tau} = \frac{1}{16}\Bigg[\Big(& 2\{F_{\tau\nu}, F_{\mu\rho}\}^{ab} e^c_\lambda - \omega^{ab}_\lambda (D_\rho F^{cd}_{\tau\mu} + \partial_\rho F^{cd}_{\tau\mu}) e^m_\nu \eta_{dm} - \{\omega_\nu, (D_\rho F_{\tau\mu} + \partial_\rho F_{\tau\mu})\}^{ab} e^c_\lambda - \partial_\tau \{\omega_\nu, (\partial_\rho \omega_\mu + F_{\rho\mu})\}^{ab} e^c_\lambda \\ & - \omega^{ab}_\lambda \partial_\tau \left(\omega^{cd}_\nu \partial_\rho e^m_\mu + \left(\partial_\rho \omega^{cd}_\mu + F^{cd}_{\rho\mu}\right) e^m_\nu\right)\eta_{dm} + 2\partial_\nu \omega^{ab}_\lambda \partial_\rho \partial_\tau e^c_\mu - 2\partial_\rho \left(\partial_\tau \omega^{ab}_\mu + F^{ab}_{\tau\mu}\right)\partial_\nu e^c_\lambda - \{\omega_\nu, (\partial_\rho \omega_\lambda + F_{\rho\lambda})\}^{ab} \partial_\tau e^c_\mu \\ & - \left(\partial_\tau \omega^{ab}_\mu + F^{ab}_{\tau\mu}\right)\left(\omega^{cd}_\nu \partial_\rho e^m_\lambda + \left(\partial_\rho \omega^{cd}_\lambda + F^{cd}_{\rho\lambda}\right) e^m_\nu\right)\eta_{dm}\Big)\eta_{cb} - \omega^{ac}_\lambda \omega^{db}_\nu e^f_\rho F^{gm}_{\tau\mu} \eta_{cd}\eta_{fg}\eta_{bm}\Bigg], \end{aligned}$$

where the brackets are defined as

$$\{\alpha, \beta\}^{ab} = \left(\alpha^{ac}\beta^{db} + \beta^{ac}\alpha^{db}\right)\eta_{cd}, \quad [\alpha, \beta]^{ab} = \left(\alpha^{ac}\beta^{db} - \beta^{ac}\alpha^{db}\right)\eta_{cd},$$

$$D_\mu F^{ab}_{\rho\sigma} = \partial_\mu F^{ab}_{\rho\sigma} + \left(\omega^{ac}_\mu F^{db}_{\rho\sigma} - \omega^{db}_\mu F^{ac}_{\rho\sigma}\right)\eta_{cd}.$$

and the real deformed metric is given by the formula [4]

$$\hat{g}_{\mu\nu} = \frac{1}{2}\left(\hat{e}^b_\mu * \hat{e}_{\nu b} + \hat{e}^b_\nu * \hat{e}_{\mu b}\right).$$

where $*$ denotes the star product

$$(f * g)(x) = f(x)\, e^{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, g(x).$$

Remark: All calculations in this work are carried out to second order in the NC parameter Θ.

RESULTS & DISCUSSION

The NC geodesic equation for massive particle in the NC Schwarzschild spacetime, in the second order in Θ, is given by:

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(E^2 - m_0^2 c^2)}{l^2} + \frac{2mm_0^2c^2}{l^2}u - u^2 + 2mu^3 + \frac{\Theta^2}{2l^2}\left\{(E^2 - m_0^2c^2)u^2 \quad + 6m(m_0^2c^2 - E^2)u^3\right\},$$

To solve this equation we use an approximation solution, to do this, let us rewrite the above equation in the following form:

$$\frac{d^2u}{d\phi^2} + \omega^2 u = \frac{m}{\tilde{l}^2} + 3m'u^2,$$

with the following changes $\tilde{E} = \frac{E_0}{m_0 c^2}$, $\tilde{l} = \frac{l}{m_0 c}$ we obtain:

$$\omega^2 = 1 - \frac{\Theta^2}{2\tilde{l}^2}(\tilde{E}^2 - 1), \quad m' = m\left[1 + 3\Theta^2\left(\frac{1 - \tilde{E}^2}{2\tilde{l}^2}\right)\right],$$

The solution of this equation can be expressed by using successive-approximation method [3], where we assume a general solution of the form $u = u_0 + u_1$, where u_0 is the solution of the first order (Newtonian equation),

$$u(\phi) = \frac{m}{\tilde{l}^2\omega^2}\left(1 + e\cos\left(\omega\phi - \frac{3m\,m'}{\tilde{l}^2\omega^3}\phi\right)\right).$$

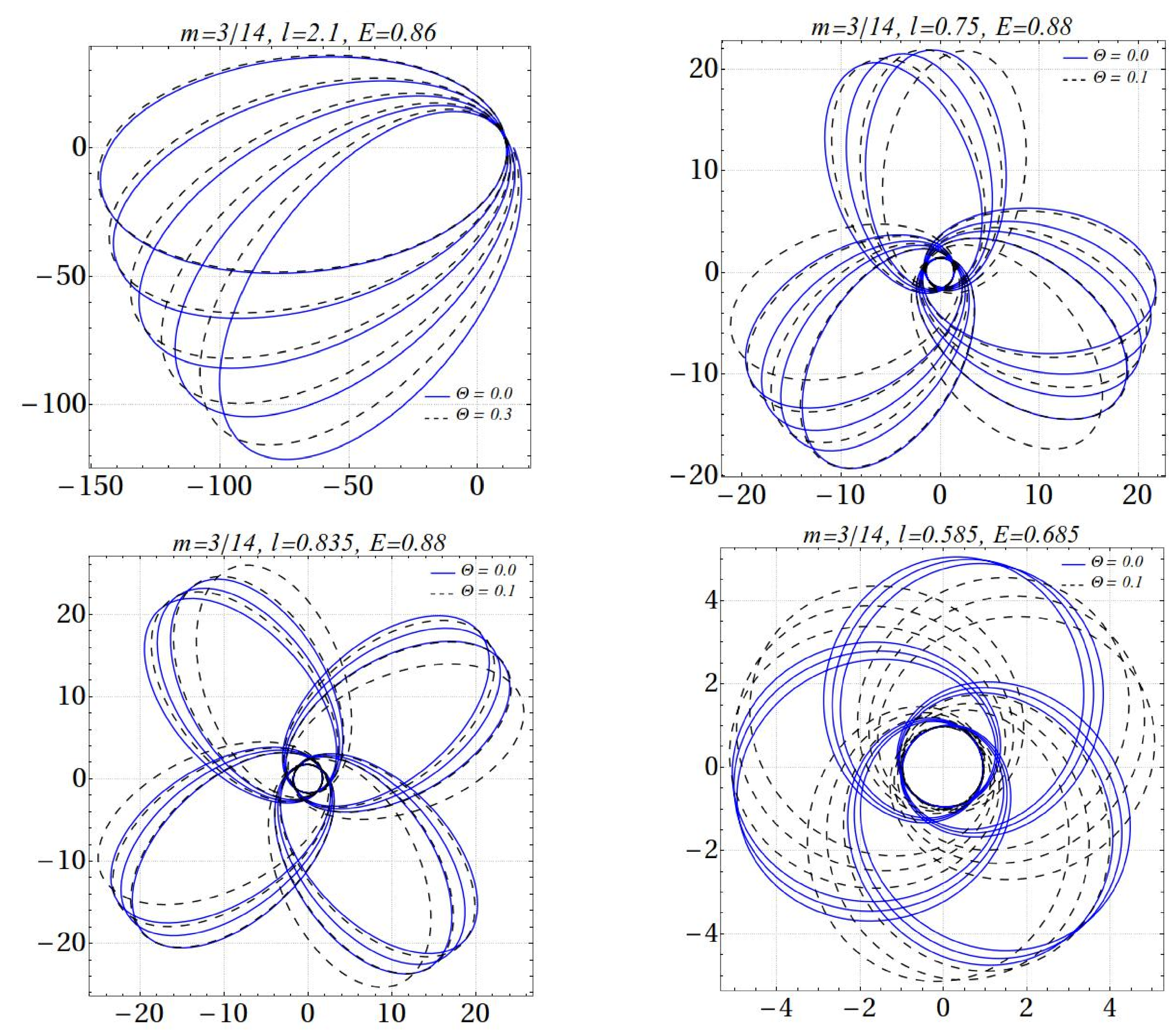

Fig. 1: Periodic orbits of neutral massive particle around Schwarzschild BH for fixed mass.

Upper bound on the NC parameter:

In order to estimate an upper bound on the NC parameter, we use the fact that the NC corrections remain smaller than the experimental accuracy [5]. However, the actual observational accuracy is approximately 1%-10% according to Event Horizon Telescope measurements [6]. In this application we choice the smaller accuracy, , and the upper bound becomes:

$$\left|\frac{\Delta\phi_{GR} - \Delta\hat{\phi}}{\Delta\phi_{GR}}\right| \leq 0.01.$$

with

$$\Delta\hat{\phi} = \frac{6\pi G\,M}{c^2\,\alpha(1 - e^2)} + \Theta^2\left\{\frac{\pi\,c^2(\tilde{E}^2 - 1)}{2G\,M\,\alpha(1 - e^2)} + \frac{9\pi(1 - \tilde{E}^2)}{\alpha^2(1 - e^2)^2}\right\},$$

As an application we use a typical primordial BH with mass $m = GM \sim 5 \times 10^{-4}\ m$ and size $r \sim 1.5 \times 10^{-3}\ m$ at the final stage of inflation [5]. By using S2-like orbit system, the physical upper limit for the NC parameter is [7]:

$$\Theta^{Phy} \leq 2.18 \times 10^{-32}\ m$$

CONCLUSION

In this present work, we obtain an analytical solution to the NC geodesic equation of neutral massive test particle around the Schwarzschild BH, using an approximation method. Then we plot some orbits and show the effect of the non-commutativity on the orbital motion of neutral massive test particles. Our results show that non-commutativity of the spacetime affects the orbital motion of the test particle, and its effect is confined to the deformation on the periastron advance of this particle's orbit.

REFERENCES

[1] Tajron Jurić, A Naveena Kumara, and Filip Požar. Nuclear Physics B, page 116950, 2025.
[2] Abdellah Touati and Slimane Zaim. Nuclear Physics B, page 117018, 2025.
[3] MI Wanas and MA Bakry. Astrophysics and Space Science, 228(1):203–220, 1995.
[4] Masud Chaichian, Anca Tureanu, and G Zet. Physics Letters B, 660(5):573–578, 2008.
[5] Majid Karimabadi, S Aliasghar Alavi, and Davood Mahdavian Yekta. Classical and Quantum Gravity, 37(8):085009, 2020.
[6] The Event Horizon Telescope Collaboration et all. The Astrophysical Journal Letters, 875(1):L1, apr 2019.
[7] Pravabati Chingangbam, Subinoy Das, et al. Physical Review D, 91(8):083503, 2015.