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Energy-Stable Variable-Order Fractional Diffusion in Heterogeneous Biological Media

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INTRODUCTION & AIM

Anomalous diffusion arises in many heterogeneous biological systems where transport deviates from classical Brownian motion.

Key mechanisms include:

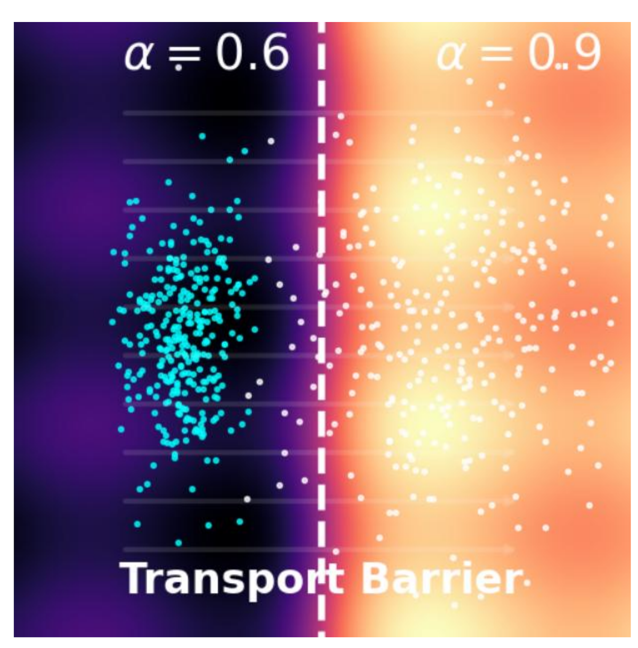
- cellular crowding
- membrane barriers
- trapping effects
- heterogeneous porosity
- long-range transport correlations

Classical diffusion models assume homogeneous transport and therefore fail to capture spatially varying propagation regimes observed in realistic tissues.

Variable-order fractional diffusion introduces a spatially adaptive exponent α = α(x), allowing the transport dynamics to adapt locally to the surrounding microstructure.

This framework naturally captures:

- heterogeneous diffusion
- subdiffusive trapping
- transport barriers
- nonlocal interactions
- multiscale propagation dynamics



METHOD

We consider the variable-order time–space fractional diffusion equation

$$ {}^{C}D_t^{\alpha_t} u(x,t) = \mathcal{L}_{\alpha(x)} u(x,t), \quad (x,t) \in \Omega \times (0,T]. $$

The Caputo derivative introduces memory effects:

$$ {}^{C}D_t^{\alpha_t} u(x,t) = \frac{1}{\Gamma(1-\alpha_t)} \int_0^t \frac{\partial u(x,s)}{\partial s} (t-s)^{-\alpha_t} ds. $$

The nonlocal operator is defined by

$$ \mathcal{L}_{\alpha(x)} u(x) = \int_{\Omega} (u(x) - u(y)) K_{\alpha(x,y)}(x,y)\, dy. $$

The kernel reads

$$ K_{\alpha}(x,y) = \frac{C(\alpha)}{|x-y|^{2+\alpha}}. $$

The spatially varying fractional order controls the local transport regime:

$$ \alpha(x) = \begin{cases} 0.6, & x < 0.5, \\ 0.9, & x \geq 0.5. \end{cases} $$

The continuous problem admits the nonlocal energy functional

$$ E(u) = \frac{1}{2} \int_{\Omega} \int_{\Omega} (u(x) - u(y))^2 K_{\alpha(x,y)}\, dx\, dy. $$

The energy satisfies the dissipative property

$$ \frac{d}{dt} E(u(t)) \leq 0, $$

ensuring irreversible relaxation toward equilibrium.

The spatial discretisation is given by

$$ (Au)_i = \sum_{j \neq i} (u_i - u_j) w_{ij}, $$

where the coefficients approximate the nonlocal interactions.

The Caputo derivative is approximated using the L1 scheme:

$$ D_t^{\alpha} u^n \approx \sum_{k=0}^{n-1} b_k (u^{n-k} - u^{n-k-1}), $$

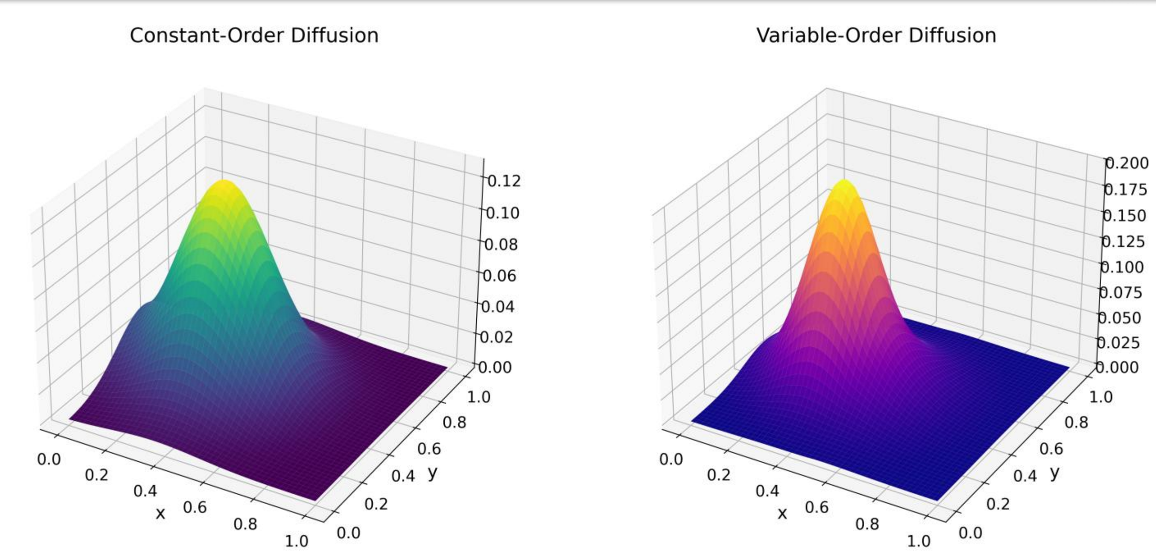
with

$$ b_k = (k+1)^{1-\alpha} - k^{1-\alpha}. $$

The fully discrete implicit scheme reads

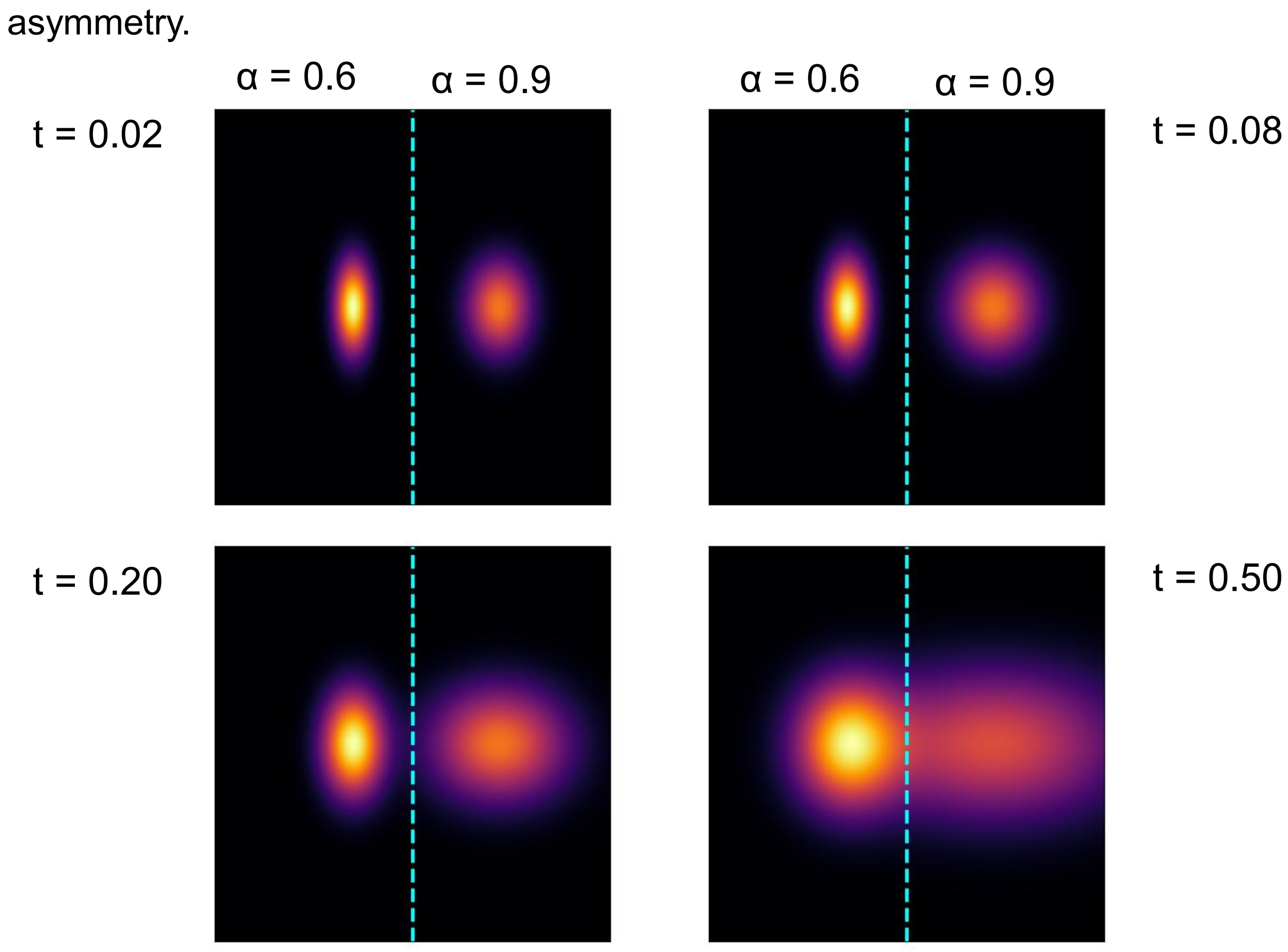
$$ (I + \Delta t A) u^{n+1} = u^n - \Delta t \sum_{k=0}^{n-1} b_k (u^{n-k} - u^{n-k-1}). $$

RESULTS & DISCUSSION

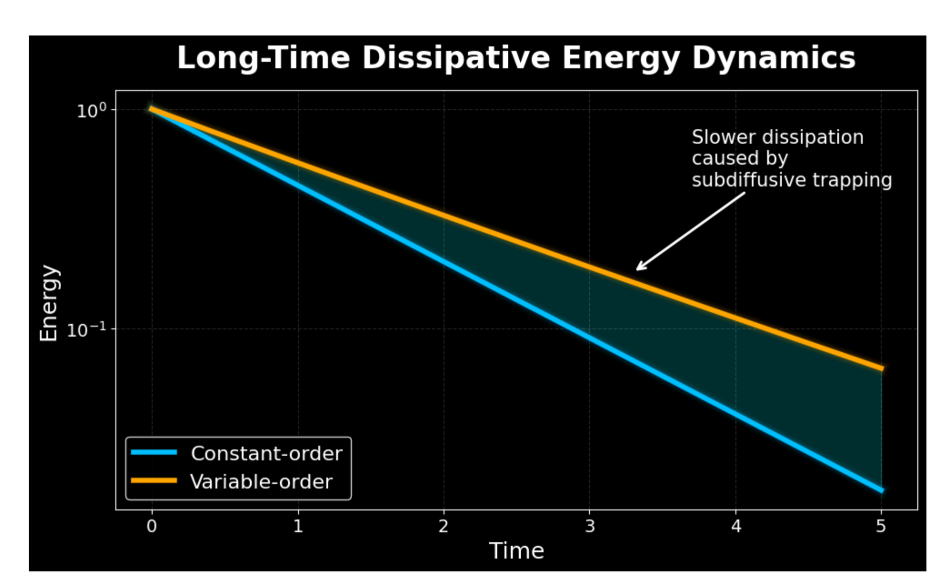


Constant-order models predict nearly symmetric transport.

Variable-order dynamics reveal trapping effects, delayed propagation, and strong interface-induced asymmetry.



The heterogeneous interface fundamentally reshapes anomalous transport dynamics. Subdiffusive regions act as effective transport barriers, producing delayed propagation, localized trapping, and strongly asymmetric spreading patterns.



The variable-order model exhibits significantly slower long-time energy dissipation due to heterogeneous subdiffusive trapping effects.
Spatial heterogeneity delays relaxation toward equilibrium and fundamentally modifies the global dissipative dynamics.

CONCLUSION

Spatial heterogeneity fundamentally modifies anomalous transport dynamics and generates propagation regimes that cannot be reproduced by classical homogeneous diffusion models.

The proposed variable-order fractional framework successfully captures subdiffusive trapping, nonlocal interactions, and interface-induced transport barriers while preserving dissipative energy dynamics through an energy-stable numerical discretisation.

These results demonstrate the importance of locally adaptive fractional models for realistic simulation of transport processes in complex heterogeneous biological media.

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