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Permanence, Stability, Extinction and Chaos in a Nonautonomous Predator–Prey Model with Additive Allee Effect

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INTRODUCTION & AIM

Context & Motivation

- **Environmental Fluctuations:** Real-world ecosystems are inherently nonautonomous. Seasonal, climatic, and environmental variations dynamically alter resources and interactions, requiring time-dependent modelling rather than constant parameters.
- **Low Density Risks:** Small populations are highly vulnerable to demographic stochasticity. Incorporating an **additive Allee effect** is crucial to represent these low-density survival risks, which can dramatically shift extinction thresholds.
- **Predation Limits:** Standard linear functional responses assume infinite predator capacity. Incorporating non-linear, saturated functional responses is essential to capture realistic biological limits and handling times.

Our Objectives

This work aims to rigorously analyse this complex interplay by:

1. Establishing analytical boundaries for species survival and predator extinction.
2. Proving the global asymptotic stability of the system's trajectories.
3. Exploring the boundaries where seasonal forcing and Allee thresholds trigger chaotic behaviour.

MATHEMATICAL MODEL

1. **The Non-Autonomous System:**

$$\begin{cases} \dfrac{d}{dt}x(t) = r(t)x(t)\left(1-\dfrac{x(t)}{K(t)}\right) - \dfrac{m(t)x(t)}{b+x(t)} - \dfrac{cx(t)y(t)}{x(t)+a} \\ \dfrac{d}{dt}y(t) = -\dfrac{ecx(t)y(t)}{x(t)+a} - dy(t) - hy(t)^2 \end{cases}$$

With initial conditions: $x(0)=x_0 > 0, y(0)=y_0 > 0$

2. **Biological Parameter Interpretation**

- x, y : Prey and predator densities
- $r(t), K(t)$: Time-dependent prey growth rate and carrying capacity
- $m(t), b$: Additive Allee intensity and shape parameter $(b < K)$
- c, a : Maximum predation rate and Holling type-II half-saturation constant
- e, d, h : Efficiency conversion, predator mortality, and intraspecific competition

3. **Additive Allee Effect Thresholds**

The biological impact on prey growth is classified qualitatively into three regimes:

No Allee effect: $m \leq \frac{rb^2}{K}$ Weak Allee effect: $\frac{rb^2}{K} < m < br$

Strong Allee effect: $br < m < \frac{r(K+b)^2}{K}$

RESULTS & DISCUSSION

1. **Positivity & Ultimate Boundedness**

- **Positivity:** The positive quadrant $\mathbb{R}^2_+$ is positively invariant. Solutions always remain ecologically meaningful $(x(t) > 0, y(t) > 0)$.
- **Ultimate Bounds:** Solutions are ultimately bounded:

$$\limsup_{t\to\infty} x(t) \leq M_1 = \frac{r^u K^u}{r^l}, \qquad \limsup_{t\to\infty} y(t) \leq M_2 = \frac{ec-d}{h} (\text{ if } ec > d)$$

2. Permanence (Species Persistence)

The system is **permanent** if these biological thresholds are met:

. **Prey persistence**: $r^l - \frac{m^u}{b} - \frac{cM_2}{a} > 0$. **Predator persistence:** $\frac{ecm_1}{m_1+a} > d$

3. Global Asymptotic Stability

Using a logarithmic Lyapunov distance $V(t) = |\ln x - \ln x_1| + |\ln y - \ln y_1|$, all positive solutions asymptotically converge if :

$$A := \frac{r^l}{K^u} - \frac{m^u}{(m_1+b)^2} - \frac{cM_2}{(m_1+a)^2} - \frac{eca}{(m_1+a)^2} > 0 \quad \text{and} \quad B := h - \frac{c}{m_1+a} > 0$$

4. Predator Extinction Criterion

If the maximum potential food intake cannot compensate for the predator's natural mortality, the predator population strictly vanishes $\left(\lim_{t\to\infty} y(t) = 0\right)$:

$$\text{Extinction Condition:} \quad \frac{ecM_1}{M_1+a} < d$$

5. Emergence of Chaos

Seasonal Forcing Setup to model periodic environmental variations, parameters are modulated in time with a phase shift ($\phi_m = \pi$) creating an antiphase Allee effect during unfavourable seasons:

$$r(t) = r(1+\varepsilon_r \sin(\omega_r t + \phi_r)) \qquad K(t) = K(1+\varepsilon_K \sin(\omega_K t + \phi_K))$$

$$m(t) = m(1+\varepsilon_m \sin(\omega_m t + \phi_m))$$

Table 2. Parameter values of the system (1) for numerical simulation.

r	K	b	a	c	e	d	h
1	10	0.9	1.9	1	0.8	0.35	0.0425

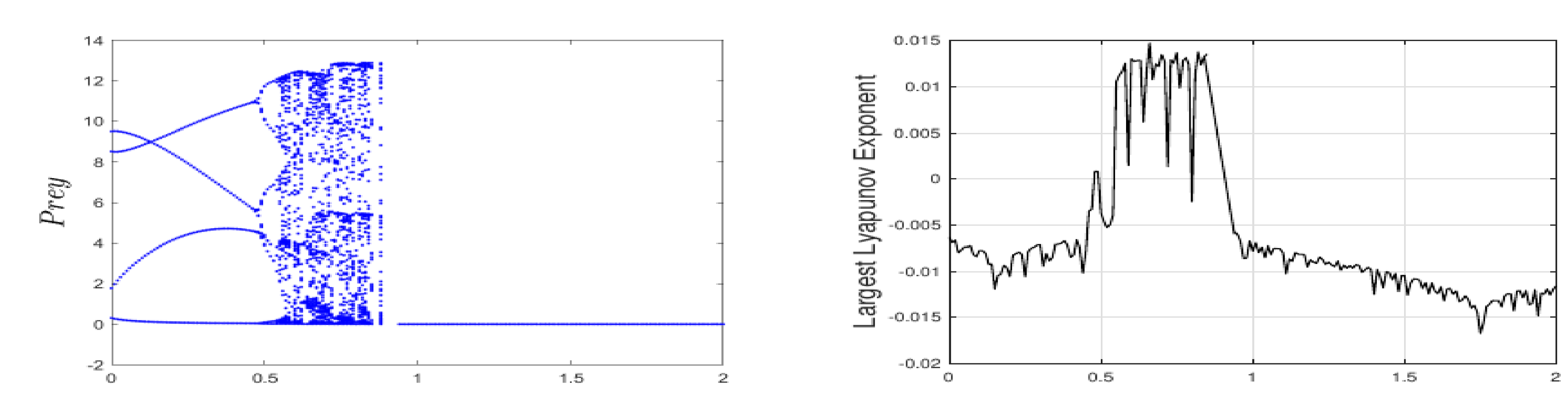

Figure 1. (a) Bifurcation diagrams of prey population showing the onset of chaos with respect to m (b) Largest Lyapunov exponent (LLE) versus m

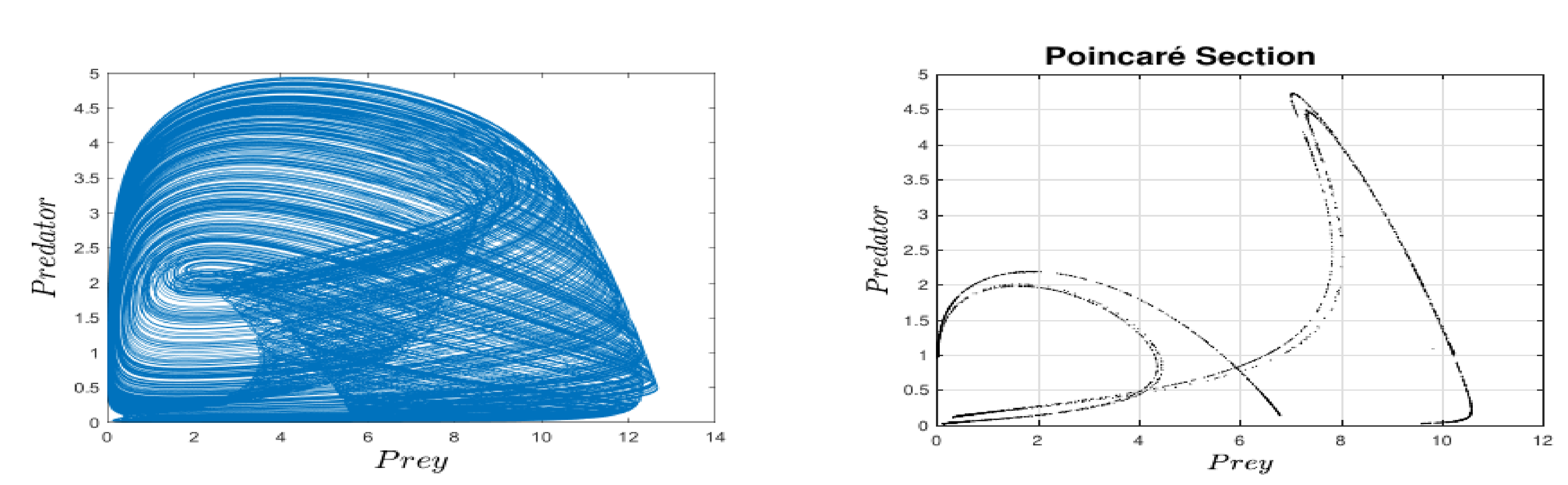

Figure 2. Phase portrait (a) and Poincaré section (b) for $m = 0.7$ illustrating the chaotic attractor of system (2).

CONCLUSION

- **Core Findings**: Established explicit analytical conditions for positivity, permanence, global stability (Lyapunov), and predator extinction in a non-autonomous system.
- **Key Discovery**: Simulations revealed that a weak additive Allee effect combined with seasonal forcing destabilizes the ecosystem, inducing chaotic oscillations.
- **Future Work:** Investigate chaos control approaches (feedback control or optimized harvesting strategies) to stabilize population fluctuations.

REFERENCES

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