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Numerical Simulation of Variable-Order Fractional Advection-Diffusion Processes with memory effects

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INTRODUCTION & AIM

What is fractional derivative?

- **Integer-order derivative (classical calculus):** $\frac{d^n y}{dt^n}, D^n y(t), y^{(n)}(t)$, where $n$is a positive integer.
- **Fractional-order derivative:** $D_t^{\alpha} u(t), 0<\alpha<1$, captures memory and nonlocal effects in complex systems.
- **Variable-order fractional derivative:** $D_t^{\alpha(t)} u(t)$ or $D_t^{\alpha(x,t)} u(x,t)$ where:
- $\alpha(t)$: time-dependent fractional order,
- $\alpha(x,t)$: space–time-dependent fractional order.

This study focuses on the following fractional advection-diffusion integrodifferential equations with variable orders in both time and space, subject to initial and boundary value conditions:

$$\begin{cases} {}_0^C D_t^{\alpha(x,t)} u(x,t) + P\dfrac{\partial u(x,t)}{\partial x} = Q\dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda\displaystyle\int_0^t K(x,t)u(x,s)ds + \mathcal{F}(x,t) \\ u(x,0) = 0, \forall x \in [0,L] \\ u(0,t) = 0, u(L,t) = 0, \forall t \in [0,T]. \end{cases}$$

P: Advection coefficient
Q: Diffusion coefficient
λ: Integral kernel parameter
$\alpha(x,t)$: Variable fractional order
$F(x,t)$: Source term

METHOD

- **Discretization:**
- **Domain discretization:**

The computational domain is discretized using uniform grids: $x_m = mh (h = L/M_x)$and $t_n = n\tau(\tau = T/N_t)$, where $M_x$and $N_t$denote the number of spatial and temporal intervals, respectively.

- **Operator discretization:**

Based on this mesh, the Caputo-type variable-order time fractional derivative is discretized as

$$ {}_0^C D_t^{\alpha(x_m,t_{n+1})} u(x_m,t_{n+1}) :\approx {}_0^C D_{N_t^m}^{\alpha_m^{n+1}} u_m^{n+1}$$

$$= \frac{\tau^{-\alpha_m^{n+1}}}{(2-\alpha_m^{n+1})}\left\{u_m^{n+1} - u_m^n + \sum_{j=1}^{n}[u_m^{n+1-j} - u_m^{n-j}][(j+1)^{1-\alpha_m^{n+1}} - j^{1-\alpha_m^{n+1}}]\right\}$$

The finite difference approximation of the first-order and second-order derivative in space is discretized as:

$\frac{\partial u(x,t)}{\partial x} =: \delta_x^1 u_m^{n+1} = \frac{u(x_{m+1},t_{n+1}) - u(x_m,t_n)}{h}$ and

$$\frac{\partial^2 u(x_m,t_{n+1})}{\partial x^2} := \delta_x^2 u_m^{n+1} = \frac{u(x_{m+1},t_{n+1}) - 2u(x,t_{n+1}) + u(x_{m-1},t_{n+1})}{h^2}$$

By using composite trapezoidal rule, The integral term is discretized as:

$$\int_0^{t_n} K(x_m,t_{n+1})u(x_m,s)ds = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} K(x_m,t_{n+1})u(x_m,s)\,ds = \frac{\tau}{2}\sum_{j=0}^{n-1}\{K(x_m,t_{n+1})u(x_m,t_{j+1}) + K(x_m,t_{n+1})u(x_m,t_j)\}$$

Using the above difference operators, the governing equation can be transformed into the following discrete form:

$$-\frac{Q\Gamma(2-\alpha_m^{n+1})\tau^{\alpha_m^{n+1}}}{h^2}u_{m-1}^{n+1} + \left(1 + 2Q\frac{\Gamma(2-\alpha_m^{n+1})\tau^{\alpha_m^{n+1}}}{h^2} - P\frac{\Gamma(2-\alpha_m^{n+1})\tau^{\alpha_m^{n+1}}}{h}\right)u_m^{n+1} + \left(\frac{P\Gamma(2-\alpha_m^{n+1})\tau^{\alpha_m^{n+1}}}{h} - \frac{Q\Gamma(2-\alpha_m^{n+1})\tau^{\alpha_m^{n+1}}}{h^2}\right)u_{m+1}^{n+1} = u_m^n - \sum_{j=1}^{n}\left[u_m^{n+1-j} - u_m^{n-j}\right]\left[(j+1)^{1-\alpha_m^{n+1}} - j^{1-\alpha_m^{n+1}}\right] + \lambda\frac{\tau^{\alpha_m^{n+1}+1}\Gamma(2-\alpha_m^{n+1})}{2}\sum_{j=0}^{n-1}\left[K(x_m,t_{n+1})u_m^{j+1} + K(x_m,t_{n+1})u_m^j\right] + \tau^{\alpha_m^{n+1}}\Gamma(2-\alpha_m^{n+1})F(x_m,t_{n+1}).$$

The discretized equations lead to a tridiagonal system, which is efficiently solved using the Thomas algorithm in MATLAB.

RESULTS & DISCUSSION

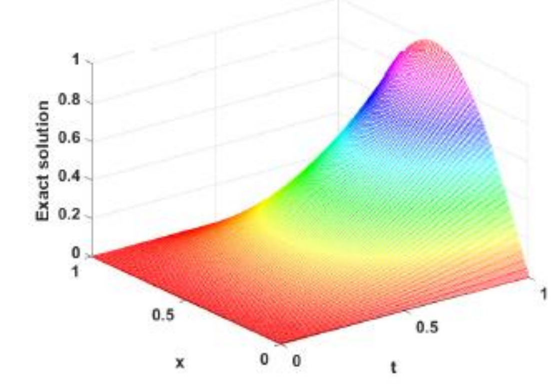
Consider $\alpha(x,t) \epsilon (0,1)$. We now examine the following VO-FPIDE

$$\begin{cases} {}_0^C D_t^{\alpha(x,t)} u(x,t) + \dfrac{\partial u(x,t)}{\partial x} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \displaystyle\int_0^t xtu(x,s)ds + \mathcal{F}(x,t) \\ u(x,0) = 0, \forall x \in [0,1] \\ u(0,t) = 0, u(1,t) = 0, \forall t \in [0,1]. \end{cases}$$

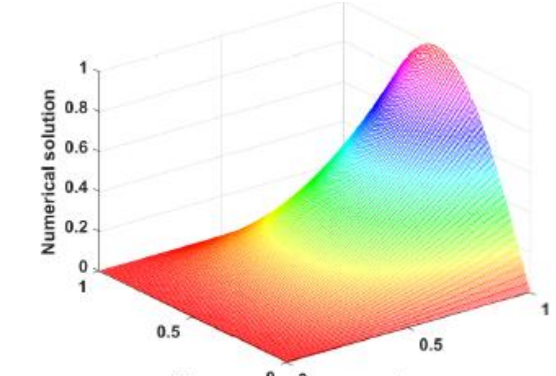
For the choice of $\mathcal{F}(x,t) = \frac{2\Gamma(2-\alpha(x,t))}{\Gamma(3-2\alpha(x,t))}t^{2-\alpha(x,t)}\sin(\pi x) + \pi t^2\cos(\pi x) + \pi^2 t^2\sin(\pi x) - \frac{xt^4}{3}\sin(\pi x)$

The exact solution is of $u(x,t) = t^2\sin(\pi x)$

α	$N_x = 10, N_t = 20$	$N_x = 20, N_t = 40$	$N_x = 40, N_t = 80$	$N_x = 80, N_t = 160$
0.2	9.5923×10^{-3}	2.1531×10^{-3}	5.1254×10^{-4}	1.2526×10^{-4}
$0.2 + 0.1t$	9.5790×10^{-3}	2.1583×10^{-3}	5.1604×10^{-4}	1.2681×10^{-4}
$0.2 + 0.1xt$	9.5836×10^{-3}	2.1544×10^{-3}	5.1374×10^{-4}	1.2582×10^{-4}
0.5	9.7168×10^{-3}	2.2398×10^{-3}	5.5184×10^{-4}	1.4139×10^{-4}
$\frac{1+e^{-0.5t}}{5}$	9.5807×10^{-3}	2.1612×10^{-3}	5.1749×10^{-4}	1.2740×10^{-4}
$\frac{1+e^{-0.5xt}}{5}$	9.5827×10^{-3}	2.1551×10^{-3}	5.1414×10^{-4}	1.2599×10^{-4}

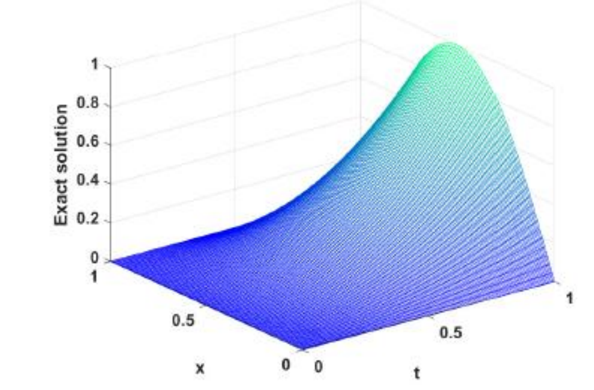


(a) Exact solution for $M_x = N_t = 20$, $\alpha = 0.5$

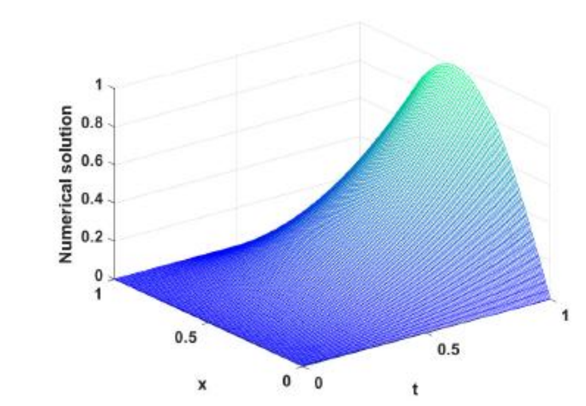


(b) Numerical solution for $M_x = 80, N_t = 160$, $\alpha = 0.5$

Fig. 1: Solution of the Example-1 with α = 0.5.

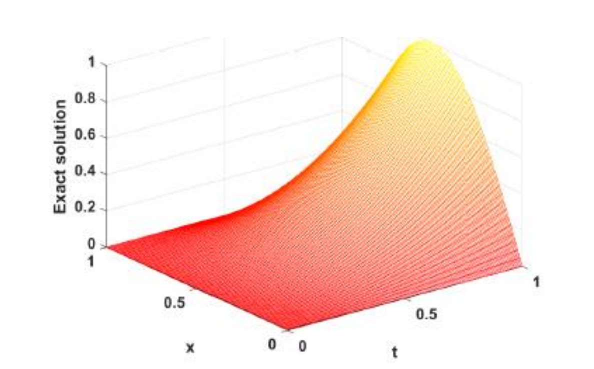


(a) Exact solution for $M_x = 80, N_t = 160$, $\alpha = \frac{1+e^{-0.5t}}{5}$.

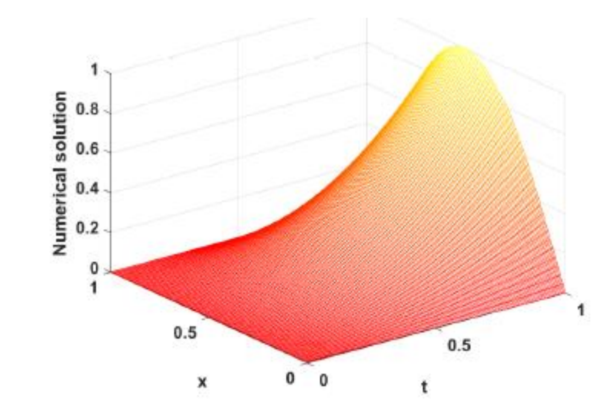


(b) Numerical solution for $M_x = 80, N_t = 160$, $\alpha = \frac{1+e^{-0.5t}}{5}$

Fig. 2: Solution of the Example-1 with $\alpha = \frac{1+e^{-0.5t}}{5}$



(a) Exact solution for $M_x = N_t = 20$, $\alpha = \frac{1+e^{-0.5xt}}{5}$.



(b) Numerical solution for $M_x = N_t = 20$, $\alpha = \frac{1+e^{-0.5xt}}{5}$.

Fig. 3: Solution of the Example-1 with $\alpha = \frac{1+e^{-0.5xt}}{5}$

CONCLUSION

In this work, we investigate a variable-order fractional advection–diffusion integrodifferential equation subject to appropriate initial and boundary conditions. A numerical scheme based on the L1 approximation, finite difference discretization, and composite trapezoidal rule is developed and validated through numerical experiments. The results demonstrate that the proposed approach accurately captures anomalous transport and memory effects, making it effective for modeling complex processes in heterogeneous media.

REFERENCES

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