IOCMA 2026 Conference

The 2nd International Online Conference on Mathematics and Applications

10–12 June 2026 | Online

Time-periodic problem for inhomogeneous heat equations in a two-layer domain with Dirichlet boundary and linear transmission conditions

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Introduction

■ Numerous works are devoted to the study of two-layer and multilayer heat conduction (diffusion) problems, where initial conditions are defined for each layer [1–3]. However, in many practical application, the system is subjected to periodic internal forcing or periodically varying environmental conditions [4]. In most cases, the classical initial conditions lose their physical meaning, since the system evolves toward a steady time-periodic regime that is independent of the initial state. Typically, the study of such problems is restricted to single-frequency harmonic perturbations or their finite superpositions.

Statement problem & Aim

■ Let $\Omega = \mathbb{R}/2\pi\mathbb{Z}$. The periodic Sobolev space $\mathbf{H}_q = \mathbf{H}_q(\Omega)$ of order $q \in \mathbb{R}$ consists of all trigonometric series $\varphi(t) = \sum_{k\in\mathbb{Z}} \hat{\varphi}_k e^{ikt}$, $\hat{\varphi}_k \in \mathbb{C}$, such that $\|\varphi\|_{\mathbf{H}_q} = \sqrt{\sum_{k\in\mathbb{Z}} (1+|k|)^{2q} |\hat{\varphi}_k|^2} < \infty$.

■ The space $\mathbf{C}^m([a,b];\mathbf{H}_q)$, where $m \in \mathbb{Z}_+$ and $[a,b] \subset \mathbb{R}$, is defined as the set of all series of the form $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x) e^{ikt}$, where $\hat{u}_k \in \mathbf{C}^m([a,b])$, and, for any fixed point $x \in [a,b]$, functions $\frac{\partial^\ell u(x,t)}{\partial x^\ell} \equiv \sum_{k\in\mathbb{Z}} \hat{u}_k^{(\ell)}(x) e^{ikt}$ belong to the space $\mathbf{H}_{q-\ell/2}$ for $\ell = 0, 1, \dots, m$, respectively, and, as the elements of this space, are continuous in x on $[a,b]$. The norm in $\mathbf{C}^m([a,b];\mathbf{H}_q)$ is defined as follows: $\|u\|_{\mathbf{C}^m([a,b];\mathbf{H}_q)} = \sum_{\ell=0}^{m} \max_{x\in[a,b]} \left\| \frac{\partial^\ell u(x,\cdot)}{\partial x^\ell} \right\|_{\mathbf{H}_{q-\ell/2}}$.

■ To formulate a problem, we partition the interval $\mathcal{D} = (x_0, x_2)$ into two subintervals $\mathcal{D}_1 = (x_0, x_1)$ and $\mathcal{D}_2 = (x_1, x_2)$. We then consider the following two-layer problem for a pair of functions $u = (u_1, u_2)$:

$$\frac{\partial u_j}{\partial t} - \alpha_j \frac{\partial^2 u_j}{\partial x^2} = f_j(x,t), \quad (x,t) \in \mathcal{D}_j \times \Omega, \quad j = 1,2, \tag{1}$$

$$u_1\big|_{x=x_0} = 0, \quad u_2\big|_{x=x_2} = 0, \quad t \in \Omega, \tag{2}$$

$$h_1 u_1\big|_{x=x_{1-}} = h_2 u_2\big|_{x=x_{1+}}, \quad \kappa_1 u_1\big|_{x=x_{1-}} = \kappa_2 u_2\big|_{x=x_{1+}}, \quad t \in \Omega, \tag{3}$$

where $\alpha_j, h_j, \kappa_j \in \mathbb{R}_+$, $\alpha_1 \neq \alpha_2$, $u_j = u_j(x,t)$, $f_j \in \mathbf{C}(\overline{\mathcal{D}}_j; \mathbf{H}_{q-1})$, $j = 1,2$.

■ This work is devoted to the analysis of the well-posedness of the problem in Sobolev spaces of time-periodic functions.

Method

■ The investigate of the problem is based on the method of separation of variables, the definition and construction of Green's functions, and estimates of the determinants related to the problem.

■ For $j = 1,2$, the Fourier series expansion in temporal harmonics

$$u_j(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_{j,k}(x) e^{ikt}, \quad f_j(x,t) = \sum_{k\in\mathbb{Z}} \hat{f}_{j,k}(x) e^{ikt}, \; x \in \overline{\mathcal{D}}_j, \tag{4}$$

reduces the problem to a countable family of the two-layer problems for ODEs with characteristic determinant $\Delta(k)$, parameterized by an integer k,

$$\Delta(k) = \begin{pmatrix} x_0 & 1 & 0 & 0 \\ 0 & 0 & x_2 & 1 \\ h_1 x_1 & h_1 & -h_2 x_1 & -h_2 \\ \kappa_1 & 0 & -\kappa_2 & 0 \end{pmatrix}, \quad k = 0,$$

$$\Delta(k) = \begin{pmatrix} e^{\beta_{1,k} x_0} & e^{-\beta_{1,k} x_0} & 0 & 0 \\ 0 & 0 & e^{\beta_{2,k} x_2} & e^{-\beta_{2,k} x_2} \\ h_1 e^{\beta_{1,k} x_1} & h_1 e^{-\beta_{1,k} x_1} & -h_2 e^{\beta_{2,k} x_1} & -h_2 e^{-\beta_{2,k} x_1} \\ \kappa_1 \beta_{1,k} e^{\beta_{1,k} x_1} & -\kappa_1 \beta_{1,k} e^{-\beta_{1,k} x_1} & -\kappa_2 \beta_{2,k} e^{\beta_{2,k} x_1} & \kappa_2 \beta_{2,k} e^{-\beta_{2,k} x_1} \end{pmatrix}, \; k \neq 0,$$

where $\beta_{j,k} = (1 + \operatorname{sgn}(k) i) \sqrt{\frac{|k|}{2\alpha_j}}$ for $k \neq 0$ and $j = 1,2$.

■ It is proved that $\Delta(k)$ is nonzero for all k.

■ For each integer k, the corresponding Green's function $G_k(x,\xi)$ is constructed, which allows one to obtain an exact solution

$$u_j(x,t) = \sum_{k\in\mathbb{Z}} \left(\int_{x_0}^{x_2} G_k(x,\xi) F_k(\xi) d\xi \right) e^{ikt}, \quad x \in \overline{\mathcal{D}}_j, \tag{5}$$

where F_k is a function on $\overline{\mathcal{D}}$ such that $F_k\big|_{\overline{\mathcal{D}}_j} = \hat{f}_{j,k}$,

$$G_k(x,\xi)\big|_{\overline{\mathcal{D}}_j} = \left(B_{2j-1,k} x + B_{2j,k}\right) + \frac{H((x_1-\xi)\operatorname{sgn}(x_1-x)) \min\{x,\xi\}}{\alpha_j}, \quad k = 0,$$

$$G_k(x,\xi)\big|_{\overline{\mathcal{D}}_j} = \left(B_{2j-1,k} e^{\beta_{j,k} x} + B_{2j,k} e^{-\beta_{j,k} x}\right) + \frac{H((x_1-\xi)\operatorname{sgn}(x_1-x))\, e^{-\beta_{j,k}|x-\xi|}}{2\alpha_j \beta_{j,k}}, \quad k \neq 0,$$

$j = 1,2$. At the points $x = x_0$, $x = x_1$ and $x = x_2$ the function $G_k(x,\xi)$ is defined in the sense of right- and left-hand limits. Here, the coefficients $B_{q,k} = B_{q,k}(\xi)$ are determined using formulas

$$B_{q,k}(\xi) = -\frac{1}{\Delta(k)} \left(\frac{H(x_1-\xi)\, U^1_{qk}(\xi)}{\alpha_1} + \frac{H(\xi-x_1)\, U^2_{qk}(\xi)}{\alpha_2} \right), \quad q = 1, \dots, 4,$$

$$U^1_{qk}(\xi) = \begin{cases} A_0^{1,q} x_0 + A_0^{3,q} h_1 \min\{x_1,\xi\}, & k = 0, \\ \frac{A_k^{1,q}}{2\beta_{1,k}} e^{-\beta_{1,k}|x_0-\xi|} + \frac{1}{2}\left(\frac{A_k^{3,q} h_1}{\beta_{1,k}} + A_k^{4,q}\kappa_1\right) e^{-\beta_{1,k}|x_1-\xi|}, & k \neq 0, \end{cases}$$

$$U^2_{qk}(\xi) = \begin{cases} A_0^{2,q} \xi - A_0^{3,q} h_2 \min\{x_1,\xi\} - A_0^{4,q}\kappa_2, & k = 0, \\ \frac{A_k^{2,q}}{2\beta_{2,k}} e^{-\beta_{2,k}|x_2-\xi|} - \frac{1}{2}\left(\frac{A_k^{3,q} h_2}{\beta_{2,k}} + A_k^{4,q}\kappa_2\right) e^{-\beta_{2,k}|x_1-\xi|}, & k \neq 0, \end{cases}$$

where $A_k^{s,q}$ is the algebraic (s,q)-cofactor of $\Delta(k)$.

Main results

Theorem 1. *If $f_j \in \mathbf{C}(\overline{\mathcal{D}}_j; \mathbf{H}_{q-1/2})$, $j = 1,2$, then problem (1)–(3) has a unique solution $u = (u_1,u_2) \in \mathbf{C}^2(\overline{\mathcal{D}}_1; \mathbf{H}_q) \times \mathbf{C}^2(\overline{\mathcal{D}}_2; \mathbf{H}_q)$. The components u_1 and u_2 are represented by the series (5) and depend continuously on the functions f_1 and f_2, i.e., $\|u_j\|_{\mathbf{C}^2(\overline{\mathcal{D}}_j;\mathbf{H}_q)} \le C_j \sum_{s=1}^{2} \|f_s\|_{\mathbf{C}(\overline{\mathcal{D}}_s;\mathbf{H}_{q-1/2})}$, where C_j is a positive constant independent of k, $j = 1,2$.*

■ Let $u_{j,N}$ denote the truncation at level N of the Fourier series u_j in (4).

Theorem 2. *Let $f_j \in \mathbf{C}(\overline{\mathcal{D}}_j; \mathbf{H}_{q-1/2+\theta})$, where $j = 1,2$, $q \in \mathbb{R}$, $\theta > 0$, and let $u = (u_1,u_2)$ be the solution of the problem (1)–(3). Then the function u_j and its approximation $u_{j,N}$ satisfy the following approximation estimate:*

$$\|u_j - u_{j,N}\|_{\mathbf{C}^2(\overline{\mathcal{D}}_j;\mathbf{H}_q)} < C_j \varrho (1+N)^{-\theta}, \quad j = 1,2,$$

where $\varrho = \sum_{s=1}^{2} \|f_s\|_{\mathbf{C}(\overline{\mathcal{D}}_s;\mathbf{H}_{q-1/2+\theta})}$ and C_j is a constant defined in Theorem 1.

Conclusion

■ Thus, a solution to the original problem has been constructed in the form of Fourier series, and conditions for the existence and uniqueness of the solution in Sobolev spaces have been established. The obtained results can used for construction the approximate solutions to the considered problem.

References

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