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# A Generalization of a Geometric Property of Blaschke Products to some Selected Riemann Surfaces

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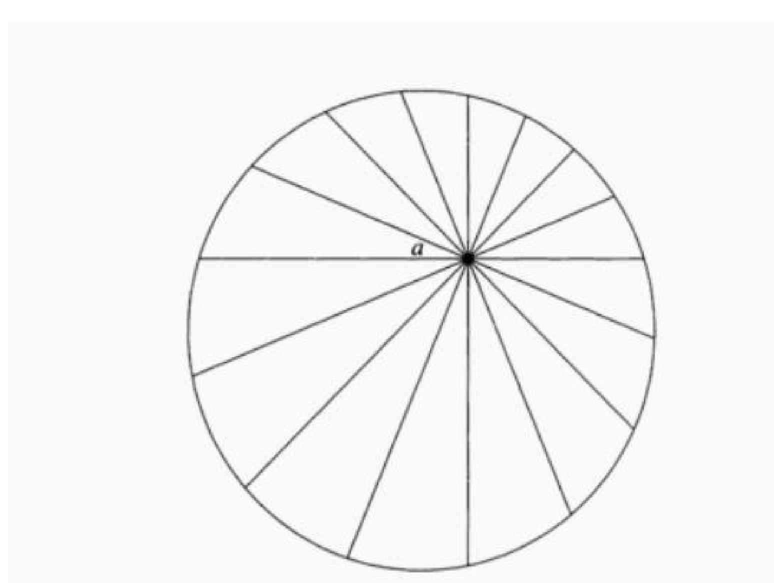
## INTRODUCTION & AIM

In [1], it has been proved that, for a Blaschke product $B(z) = z\left(\frac{z-a}{1-\bar{a}z}\right), a \in \mathbb{D}$, any two points on the unit circle with the same image lie on a straight line passing through a fixed point in the unit disc. This geometric property serves as the foundation of the study and the goal is to extend it to more general analytic settings and Riemann surfaces.

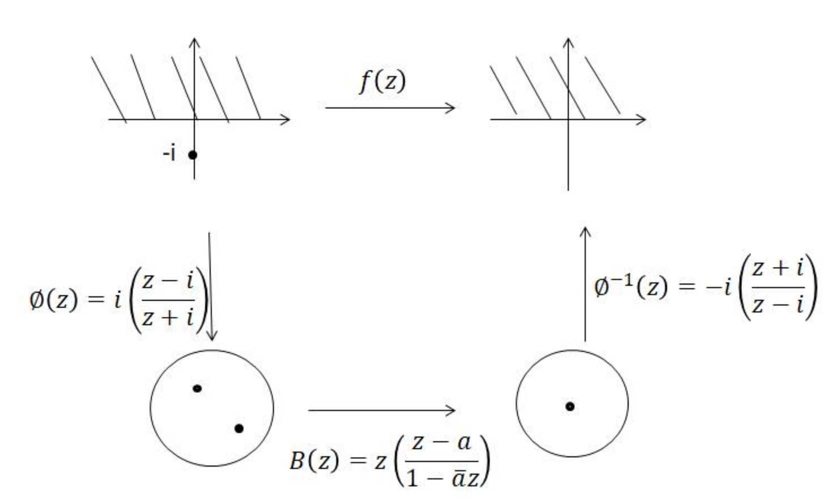
**AIMS**

- Study the collinearity property of degree–2 Blaschke products
- Extend the results to rational Nevanlinna functions on the upper half–plane
- Construct a function on the compact Riemann surface of $\sqrt{z}$ which exhibits a similar geometric property.



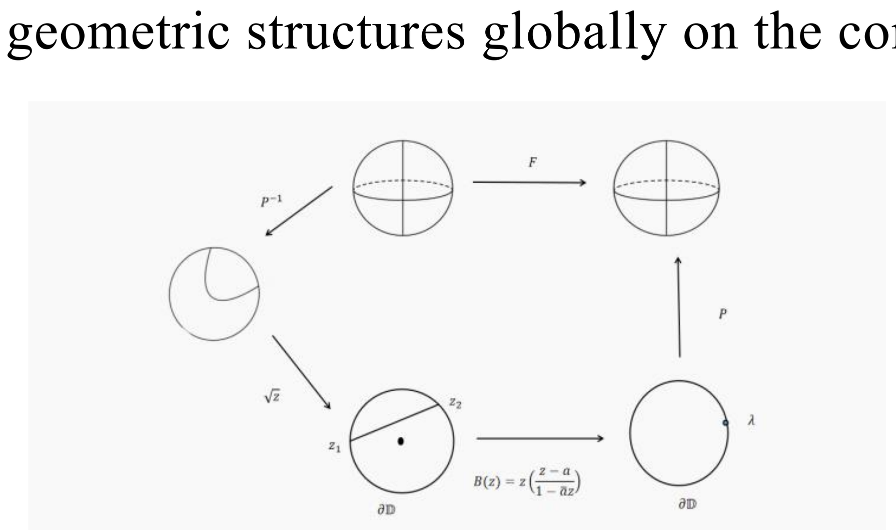

## METHOD

The study begins with the well-known geometric property of degree-2 Blaschke products in the unit disc $\mathbb{D}$ and transfers it to rational Nevanlinna function on the upper half-plane $\mathbb{H}$ using a sequence of conformal maps.




Then, the same geometric property of Blaschke products is further lifted to the compact Riemann surface of $\sqrt{z}$ through the covering transformation $z = w^2$.Under the squaring map, chords of the unit disc transform into parabolas, which lift to spherical parabolic curves on the compact Riemann surface; here *spherical parabolas* mean closed parabolic curves intersecting at infinity.

In particular, the pre–image of a circular arc orthogonal to the unit circle becomes a parabola on the Riemann surface. This construction connects the unit disc and upper half–plane geometries with the global structure of the Riemann sphere, providing a unified framework for visualizing the induced analytic and spherical geometric structures globally on the compact Riemann surface.




## RESULTS & DISCUSSION

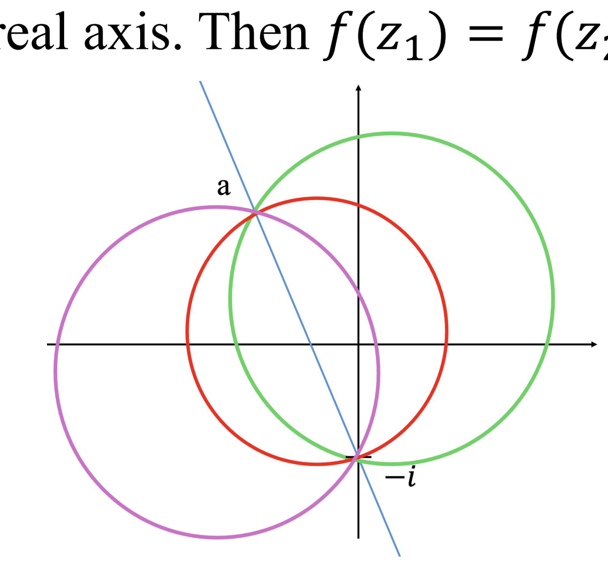
**Theorem 1**

Let $f(z) = A + \frac{B_1}{z-r_1} + \frac{B_2}{z-r_2}$ where $A, B_1, B_2, r_1, r_2 \in \mathbb{R}, B_1, B_2 < 0$ and $r_1 \neq r_2$ be a rational Nevanlinna function. Then, there exists a fixed point $\alpha \in \mathbb{H}$,

which we called the base point of $f$, such that the following properties hold:

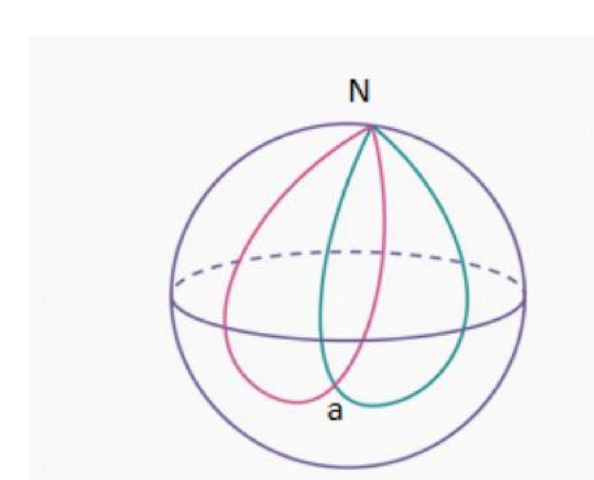
- for any $\lambda \in \mathbb{R} \setminus \{A\}$, the equation $f(z) = \lambda$ has two distinct real solutions $z_1$ and $z_2$, and the four points $z_1, z_2, \alpha$, and $-i$ lie on a common circle.
- The points $r_1, r_2, \alpha$ and $-i$ are concyclic.
- The three points $\frac{B_1 r_2 + B_2 r_1}{B_1 + B_2}$, $\alpha$, $-i$ are collinear.

Conversly,
Let $\alpha$ be the base point of $f$ (as describe in the above part) in the upper half-plane $\mathbb{H}$. Let C be any circle passing through $\alpha$ and $-i$, and let $z_1, z_2$ be the intersections of C with the real axis. Then $f(z_1) = f(z_2)$.




**Theorem 2**

Let $w_1$ and $w_2$ be two distinct points on the equator of the Riemann sphere that are not diametrically opposite. Then there exists a unique spherical parabola on the compact Riemann surface of $\sqrt{z}$ passing through $w_1$,$w_2$ and the point at infinity. Moreover, any two such spherical parabolas intersect at a fixed point in the lower hemisphere.




## CONCLUSION

This study shows how a perticular geometric property of degree 2 Blaschke products transform under conformal mappings. Starting from case of closed unit disk, it extends to concyclicity in the upper half–plane and spherical curves on the Riemann surface via the squaring map. The sequence line segents, circles, and spherical curves reflect a link between complex analysis and geometry. Overall, it highlights the role of Riemann surfaces in visualizing multi-valued functions.

## REFERENCES

1. Daep, U., Gorkin, P., & Mortini, R. (2002). Ellipses and finite Blaschke products. Journal of Algebra, 109 (2), 785–795.
2. Egahi, M., Agbata, B. C., Ogwuche, O. I., & Soomiyol, M. C. (2020). Extended complex plane and Riemann sphere. Scientific Research Journal (SCIRJ), 8 (4).