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On Inner Radii Estimates for Mutually Non-overlapping Domains via Green's Functions

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Preliminaries

The paper is devoted to the investigation of the problems in the geometric function theory of a complex variable. A lot of such problems are reduced to determining the maximum of the product of inner radii for a system of non-overlapping domains satisfying certain conditions.
Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0, \infty)$, $U = \{z : |z| < 1\}$ is the open unit disk.
Let a function $f(z)$, regular or meromorphic in the disk $|z| < 1$, map univalently it onto a simply connected domain $B \subset \overline{\mathbb{C}}$ such that $f(0) = a$, where $a \in B$. Then, the value

$$R(B, a) = |f'(0)|$$

is called the conformal radius of the domain B relative to a point $a \in B$. The conformal radius of the domain B with respect to infinity is $R(B, \infty) = R(\varphi(B), 0)$, where $\varphi(z) = 1/z$. An analogue of the concept of the conformal radius for a multiply connected domain is the concept of the inner radius.
Let $r(B, a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$. The inner radius of the domain B is connected with the generalized Green function $g_B(z, a)$ of the domain B by the relations

$$g_B(z, a) = \ln \frac{1}{|z-a|} + \ln r(B, a) + o(1), \quad z \to a,$$

$$g_B(z, \infty) = \ln |z| + \ln r(B, \infty) + o(1), \quad z \to \infty.$$

In 1934, Lavrentiev solved the problem on the maximum of the product of conformal radii for two non-overlapping simply connected domains:

$$R(B_1, a_1)R(B_2, a_2) \leqslant |a_1 - a_2|^2 .$$

In 1951, Goluzin obtained an accurate estimate for $n = 3$:

$$\prod_{k=1}^{3} r(B_k, a_k) \leqslant \frac{64}{81\sqrt{3}} |a_2 - a_1| \, |a_3 - a_1| \, |a_3 - a_1| .$$

In 1980, Kuzmina showed that the problem of evaluating the maximum of the product of inner radii of four domains is reduced to the smallest capacity problem in a certain continuum family and obtained the exact inequality:

$$\prod_{k=1}^{4} r(B_k, a_k) \leqslant \frac{9}{4^{8/3}} \left(\prod_{1\leqslant k<p\leqslant 4} |a_k - a_p| \right)^{2/3} .$$

No other ultimate results in this problem for $n \geqslant 5$ are known at present. In 2021, Bakhtin and Zabolotnyi obtained for $n \geqslant 2$ an effective upper estimate:

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1\leqslant p<k\leqslant n} |a_p - a_k| \right)^{\frac{2}{n-1}} .$$

Method

Let $n \in \mathbb{N}$ and let $A_n = \{a_k\}_{k=1}^{n}$ be an arbitrary fixed system of points of the complex plane $\mathbb{C}\backslash\{0\}$; $\{B_k\}_{k=0}^{n}$ be an arbitrary system of domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, and $B_i \bigcap B_j = \emptyset$, $0 \leqslant i, j \leqslant n$, $i \neq j$.
Let $d(E)$ be the transfinite diameter of a compact set $E \subset \mathbb{C}$. It is known [1] that the logarithmic capacity cap$E$ is equal to the transfinite diameter $d(E)$ of the set E. Since the inner radius of a domain containing the point at infinity is equal to the reciprocal of the transfinite diameter of the complement of said domain, we have:

$$r\left(B_0, 0\right) = r\left(B_0^+, \infty\right) = \frac{1}{d\left(\overline{\mathbb{C}} \setminus B_0^+\right)},$$

where $B^+ = \left\{z : \frac{1}{z} \in B\right\}$. According to Polya's theorem [1, 2]: $\mu E \leqslant \pi d^2(E)$, where μE is the Lebesgue measure of the compact set E; hence, we have:

$$d(E) \geqslant \left(\frac{1}{\pi}\mu E\right)^{\frac{1}{2}} .$$

Furthermore,

$$\frac{1}{d\left(\overline{\mathbb{C}} \setminus B_0^+\right)} \leqslant \frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\overline{\mathbb{C}} \setminus B_0^+\right)}}.$$

Taking into account the monotonicity and additivity of the Lebesgue measure, we obtain:

$$\frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\overline{\mathbb{C}} \setminus B_0^+\right)}} \leqslant \frac{1}{\sqrt{\frac{1}{\pi}\mu\left(\bigcup\limits_{k=1}^{n} \overline{B}_k^+\right)}} = \frac{1}{\sqrt{\frac{1}{\pi}\sum\limits_{k=1}^{n} \mu\overline{B}_k^+}}.$$

It follows from the area minimization theorem [1] that $\mu(B) \geqslant \pi r^2\left(B, a\right)$ and thus

$$r\left(B_0, 0\right) \leqslant \left(\sum_{k=1}^{n} r^2\left(B_k^+, a_k^+\right)\right)^{-\frac{1}{2}} .$$

Using the invariance of the Green function under conformal and univalent mappings, we have:

$$r\left(B_k^+, a_k^+\right) = \frac{r\left(B_k, a_k\right)}{|a_k|^2}.$$

The following inequality, which is essential for proving the main results of this work, holds:

$$r\left(B_0, 0\right) \leqslant \left(\sum_{k=1}^{n} \frac{r^2\left(B_k, a_k\right)}{|a_k|^4}\right)^{-\frac{1}{2}} .$$

Estimates of the products of the inner radii of domains

This work is devoted to obtaining effective upper bounds for functionals of the following form for all values of the parameter $\gamma \in \mathbb{R}^+$:

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $n \in \mathbb{N}$, $A_n = \{a_k\}_{k=1}^{n}$ is an arbitrary fixed system of points in the complex plane $\mathbb{C}\backslash\{0\}$, and B_0, B_∞, $\{B_k\}_{k=1}^{n}$ is a system of pairwise non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, and $a_k \in B_k \subset \overline{\mathbb{C}}$ for $k = \overline{1, n}$. The following results hold.
Theorem 1. Let $n \in \mathbb{N}$, $\gamma \in \mathbb{R}^+$. Then, for any set of the points $\{a_k\}$, $k = \overline{1, n}$, on the unit circle $|z| = 1$ and for an arbitrary set of domains $\{B_k\}$, $k = \overline{0, n}$, such that $a_k \in B_k \subset \mathbb{C}$, $k = \overline{0, n}$, $a_0 = 0$, $B_i \cap B_j = \emptyset$, $i \neq j$, the following dynamic inequality holds:

$$I_n(\gamma) \leqslant n^{-\frac{\gamma}{2}} \left(I_n(0)\right)^{1-\frac{\gamma}{n}} .$$

Furthermore, since

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) = r^{\tau}\left(B_0, 0\right) \left(r^{\gamma-\tau}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \right),$$

then we obtain the following statement for the functional $I_n(\gamma)$.
Theorem 2. Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0,\, n]$, and $\tau \in (0,\, \gamma)$. Then, for any set of the points $\{a_k\}$, $k = \overline{1, n}$, on the unit circle $|z| = 1$ and for an arbitrary set of domains $\{B_k\}$, $k = \overline{0, n}$, such that $a_k \in B_k \subset \mathbb{C}$, $k = \overline{0, n}$, $a_0 = 0$, $B_i \cap B_j = \emptyset$, $i \neq j$, the following evolutionary-type inequality holds:

$$I_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} I_n(\tau) \left(I_n(0)\right)^{-\frac{\gamma-\tau}{n}} .$$

The quantity $h(z, a) = \ln r(B, a)$, which appears as the constant term in the expansion of the Green function is called the Robin constant (or the logarithmic potential of the equilibrium measure) of the domain B with respect to the point a.
Theorem 3. Let $n \in \mathbb{N}$, $n \geqslant 2$; $a_k \in \mathbb{C}$, $k = \overline{1, n}$, be a set of points on the circle $|z| = 1$; and f_k, $k = \overline{1, n}$, be functions that are regular and univalent in the unit disk $|z| < 1$, and map the disk onto mutually non-overlapping domains such that $f_k(0) = a_k$, $f_k(U) = B_k$, and $f_i(z_1) \neq f_j(z_2)$ for arbitrary natural numbers $1 \leqslant i, j \leqslant n$, $i \neq j$, and arbitrary distinct $z_1, z_2 \in U$. Then, the following inequality holds:

$$\sum_{k=0}^{n} h_k(z, a_k) \leqslant -\frac{n-1}{4} \ln\left(n-1\right) - \frac{1}{2} \ln n + \frac{2}{n} \ln \left(\prod_{1\leqslant p<k\leqslant n} |a_p - a_k| \right) .$$

Inequality for derivatives of rational functions

In this work, we propose a solution to the relevant problem of establishing multi-point estimates for a class of rational functions whose critical values are concentrated on the segment $[a, b]$. The scientific novelty of the proposed approach lies in the use of metric characteristics of the mutual arrangement of zeros (or poles), which allows not only for the refinement of known results but also for the investigation of distortion dynamics depending on the geometric configuration of points. Such an approach opens new possibilities for the analysis of functions under conditions of local concentration of their zeros, where traditional estimates often prove to be too coarse.
Theorem 4. Let f be a rational function of degree $n \geqslant 2$, such that all its critical values belong to the interval $[a, b]$, where $0 < a < b < \infty$. Let z_k, $k = 1, \ldots, n$, be the zeros of this function ($f(z_k) = 0$) located on the unit circle $|z| = 1$. Then, the following inequality holds:

$$\sqrt[n]{\prod_{k=1}^{n} |f'(z_k)|} \geqslant \frac{4ab}{b-a} \frac{\sqrt[4]{n-1}}{\left(\prod\limits_{1\leqslant p<k\leqslant n} |z_p - z_k| \right)^{\frac{2}{n(n-1)}}}. \tag{1}$$

Traditional Bernstein-type inequalities for rational functions typically estimate the maximum modulus of the derivative on a specific set (e.g., the unit circle) in terms of the norm of the function itself. In contrast, the resulting inequality is multi-point, as it directly relates the derivative values at all zeros (poles) simultaneously.

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