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On the (p,q)-compactness of Banach-valued Bloch mappings

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INTRODUCTION & AIM

For each $p \in (1,\infty)$, the p-convex hull of a sequence $(x_n) \in \ell_p(X)$ is defined by

$$p\text{-conv}(x_n) = \{\textstyle\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_{p^*}}\}.$$

Moreover, for the cases $p = 1$ and $p = \infty$, the 1-convex hull of a sequence $(x_n) \in \ell_1(X)$ and the ∞-convex hull of a sequence $(x_n) \in c_0(X)$ are given by

$$1\text{-conv}(x_n) = \{\textstyle\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0}\},$$

$$\infty\text{-conv}(x_n) = \{\textstyle\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1}\}.$$

According to [1], given $p \in [1,\infty]$ and $q \in [1,p^*]$, a subset $K \subseteq X$ is said to be relatively (p,q)-compact if there is a sequence $(x_n) \in \ell_p(X)$ (or $c_0(X)$ for $p=\infty$) such that $K \subseteq q^*$-conv(x_n). Thus, a bounded linear operator $T \in \mathcal{L}(X,Y)$ is called (p,q)-compact if it maps the closed unit ball of X, denoted B_X, into a relatively (p,q)-compact subset of Y. Analogously, the measure of the size of (p,q)-compactness of K is given as

$$m_{(p,q)}(K) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(X), K \subseteq q^*-\text{conv}(x_n)\} & p \in [1,\infty), \\ \inf\{\|(x_n)\|_\infty : (x_n) \in c_0(X), K \subseteq \infty-\text{conv}(x_n)\} & p = 1. \end{cases}$$

Hence, K is relatively (p,q)-compact if and only if $m_{(p,q)}(K) < \infty$. It is important to note that $(\infty,1)$-compactness agrees with the usual compactness and (p,p^*)-compactness coincides with the notion of p-compactness introduced by Sinha and Karn.

Our aim in this work is to adress the notion of (p,q)-compact linear operator to the nonlinear Bloch setting making a complete study. Let us recall that a holomorphic map $f\colon \mathbb{D} \to X$ is said to be Bloch if $\rho_{\mathcal{B}}(f) = sup_{z\in\mathbb{D}}(1-|z|^2)\|f'(z)\| < \infty$. The space of all X-valued Bloch mappings defined on $\mathbb{D}$ and such that $f(0)=0$ is denoted as $\widehat{\mathcal{B}}(\mathbb{D},X)$, and it is a Banach space endowed with the Bloch norm $\rho_{\mathcal{B}}$.

METHOD

Our main tool for developing this work will be the following theorem from [3], which establishes the theory relating to Bloch spaces.

Theorem 1. Let $z \in \mathbb{D}$.

i. A Bloch atom of $\mathbb{D}$ is a functional $\gamma_z \in \widehat{\mathcal{B}}(\mathbb{D})^*$ given by $z \mapsto f'(z)$ and such that $\|\gamma_z\| = 1/(1-|z|^2)$.
ii. The Bloch-free space over $\mathbb{D}$, denoted $\mathcal{G}(\mathbb{D})$, is defined as the norm-closed linear hull of the set $\{\gamma_z : z \in \mathbb{D}\}$.
iii. The mapping $\Gamma\colon \mathbb{D} \to \mathcal{G}(\mathbb{D})$ defined as $z \mapsto \gamma_z$ is holomorphic with $\Gamma'(z)(f) = f''(z)$ for all $f \in \widehat{\mathcal{B}}(\mathbb{D})$ and $z \in \mathbb{D}$.
iv. The space $\widehat{\mathcal{B}}(\mathbb{D})$ is isometrically isomorphic to $\mathcal{G}(\mathbb{D})^*$ via the map $\Lambda\colon \widehat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$ defined by
$$\Lambda(f)(\gamma) = \sum_{k=1}^{n} \lambda_k f'(z_k) \qquad \forall f \in \widehat{\mathcal{B}}(\mathbb{D}), \forall \gamma = \sum_{k=1}^{n} \lambda_k \gamma_{z_k} \in lin(\Gamma(\mathbb{D})).$$
v. For every complex Banach space X and every $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$, there is a unique operator $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}),X)$ such that $f' = S_f \circ \Gamma$. Furthermore, $\|S_f\| = \rho_{\mathcal{B}}(f)$. Thus, the mapping $f \mapsto S_f$ is an isometric isomorphism from $\widehat{\mathcal{B}}(\mathbb{D},X)$ onto $\mathcal{L}(\mathcal{G}(\mathbb{D}),X)$.
vi. Given $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$, we define $f^t\colon X^* \to \widehat{\mathcal{B}}(\mathbb{D})$ as $x^* \mapsto x^* \circ f$. It is called the Bloch transpose of f and it is satisfied that $f^t \in \mathcal{L}(X^*,\widehat{\mathcal{B}}(\mathbb{D}))$ with $\|f^t\| = \rho_{\mathcal{B}}(f)$. Furthermore, $f^t = \Lambda^{-1} \circ S_f^*$.

On the other hand, the theory of operator ideals is adressed to the Bloch framework through the following definition.

Definiton 2. Let $s \in (0,1]$. A s-Banach normalized Bloch ideal $[\mathcal{I}^{\widehat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}]$ is a subclass $\mathcal{I}^{\widehat{\mathcal{B}}}$ of the collection $\widehat{\mathcal{B}}$ endowed with a norm $\|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$ such that for every complex Banach space X:

i. The space $(\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},X), \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}})$ is s-Banach with $\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \geq \rho_{\mathcal{B}}(f)$ for all $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$.
ii. For all $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$, the mapping $g \cdot x$ defined as $z \mapsto g'(z)x$ belongs to the space $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$ with $\|g \cdot x\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} = \rho_{\mathcal{B}}(g)\|x\|$.
iii. The ideal property: for any $T \in \mathcal{L}(X,Y)$, $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$ and $h\colon \mathbb{D} \to \mathbb{D}$ holomorphic with $h(0)=0$, then $T \circ f \circ h \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},Y)$ with $\|T \circ f \circ h\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \leq \|T\|\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$.

A s-Banach normalized Bloch ideal $[\mathcal{I}^{\widehat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}]$ is said to be:

(R) Regular if given $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$, then $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$ if and only if $\kappa_X \circ f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D},X^{**})$ with $\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} = \|\kappa_X \circ f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$, where κ_X is the canonical isometric linear embedding from X to X^{**}.

RESULTS

For a holomorphic mapping $f\colon \mathbb{D} \to X$, we define $rang_{\mathcal{B}}(f) = \{(1-|z|^2)f'(z) : z \in \mathbb{D}\} \subseteq X$. Thus, we can extend the (p,q)-compactness to the Bloch setting as follows.

Definition 3. Let $p \in [1,\infty]$ and $q \in [1,p^*]$. A holomorphic mapping $f\colon \mathbb{D} \to X$ is said to be (p,q)-compact Bloch if $rang_{\mathcal{B}}(f)$ is a relatively (p,q)-compact subset of X. The set of all zero-preserving (p,q)-compact Bloch mappings from $\mathbb{D}$ into X is denoted by $\widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X)$. In addition, we endow it with the following norm:

$$k_{(p,q)}^{\mathcal{B}}(f) = m_{(p,q)}(rang_{\mathcal{B}}(f)) \qquad \forall f \in \widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X).$$

As a first result, we are able to show that this definition is the correct in the following sense.

Theorem 4. Let $p \in [1,\infty)$, let $q \in [1,p^*)$ and consider $s = \frac{pq}{p+q}$. Then $[\widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}, k_{(p,q)}^{\mathcal{B}}]$ is a surjective s-Banach normalized Bloch ideal, and it becomes regular for the collection of all reflexive complex Banach spaces.

Next, we characterize (p,q)-compact Bloch mappings through the property of (p,q)-compactness of its associated bounded linear operator $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}),X)$.

Theorem 5. Let $p \in [1,\infty)$, let $q \in [1,p^*)$ and let $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$. The following statements are equivalent:

i. The map $f \in \widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X)$.
ii. Its linearization $S_f \in \mathcal{K}_{(p,q)}(\mathcal{G}(\mathbb{D}),X)$.

In such a case, the correspondence $f \mapsto S_f$ is an isometric isomorphism between the spaces $(\widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X), k_{(p,q)}^{\mathcal{B}})$ and $(\mathcal{K}_{(p,q)}(\mathcal{G}(\mathbb{D}),X), k_{(p,q)})$.

The previous result allows us to state the following factorization theorem of (p,q)-compact Bloch mappings.

Corollary 6. Let $p \in [1,\infty)$, let $q \in [1,p^*)$ and let $f \in \widehat{\mathcal{B}}(\mathbb{D},X)$. The following statements are equivalent:

i. The map $f \in \widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X)$.
ii. There exist $g \in \widehat{\mathcal{B}}(\mathbb{D},Y)$ and $S \in \mathcal{K}_{(p,q)}(Y,X)$ such that $f = S \circ g$.

In this case, $k_{(p,q)}^{\mathcal{B}}(f) = \inf\{k_{(p,q)}(T)\rho_{\mathcal{B}}(g)\}$, with the infimum being taken over all such factorizations of f as above.

Defintion 7. Let $t,u,v \in [1,\infty]$ with $1 + \frac{1}{t} \geq \frac{1}{u} + \frac{1}{v}$. A holomorphic map $f\colon \mathbb{D} \to X$ is called (t,u,v)-nuclear Bloch if it can be written as $f = S \circ M_\lambda \circ g$, where $T \in \mathcal{L}(\ell_u,X)$, $M_\lambda \in \mathcal{L}(\ell_{v^*},\ell_u)$ is a diagonal operator and $g \in \widehat{\mathcal{B}}(\mathbb{D},\ell_{v^*})$. The space of all (t,u,v)-nuclear Bloch maps from $\mathbb{D}$ into X such that $f(0)=0$ is denoted by $\mathcal{N}_{(t,u,v)}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$. We endow it with the norm $\nu_{(t,u,v)}^{\mathcal{B}}(f) = \inf\{\|T\|\|M_\lambda\|\rho_{\mathcal{B}}(g)\}$ for all $f \in \mathcal{N}_{(t,u,v)}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$, where the infimum is taken over all such factorizations of f.

The preceding class of Bloch mappings has been deeply studied in [2], and we show in the next result that it is closely related to the ideal of (p,q)-compact Bloch maps.

Corollary 8. Let $p \in [1,\infty)$ and let $q \in [1,p^*)$. Then $\mathcal{N}_{(p,1,q^*)}^{\widehat{\mathcal{B}}}(\mathbb{D},X) \subseteq \widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X)$ and $k_{(p,q)}^{\mathcal{B}}(f) \leq \nu_{(p,1,q^*)}^{\mathcal{B}}(f)$ for all $f \in \mathcal{N}_{(p,1,q^*)}^{\widehat{\mathcal{B}}}(\mathbb{D},X)$.

Finally, we can characterize (p,q)-compact Bloch mappings in terms of their transposes.

Theorem 9. Let $p \in [1,\infty)$ and let $q \in [1,p^*)$. Then $f \in \widehat{\mathcal{B}}_{\mathcal{K}_{(p,q)}}(\mathbb{D},X)$ if and only if $f^t \in \mathcal{N}_{(p,q^*,1)}^{inj}(\mathcal{G}(\mathbb{D}),X)$. In such a case, $k_{(p,q)}^{\mathcal{B}}(f) = \|f^t\|_{\mathcal{N}_{(p,q^*,1)}^{inj}}$.

CONCLUSION

We are in a position to conclude that (p,q)-compact Bloch mappings extend to the nonlinear framework the classical notion of (p,q)-compact linear operator introduced by Ain, Lillemets and Oja in [1]. Thanks to this new notion, we have been able to develop a satisfactory theory on Banach normalized Bloch ideals, establishing linearization and factorization theorems, and relating this subclass with other well-known Banach normalized Bloch ideals.

REFERENCES

[1] K. Ain, R. Lillemets and E. Oja, Compact operators which are defined by ℓ_p-spaces, Quaest. Math. **35** (2012), 145--159.
[2] A. Belacel, A. Bougoutaia and A. Jiménez-Vargas, On (p,r,s)-summing Bloch maps and Lapresté norms, Adv. Oper. Theory **9** (2024), article no. 76.
[3] A. Jiménez-Vargas and D. Ruiz-Casternado, Compact Bloch mappings on the complex unit disc. arXiv:2308.02461