Global Existence Results of Weak Solutions for a Class of Abstract Evolution Problems

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1. Abstract

We prove global existence of weak solutions for a coupled reaction-diffusion system:

$$u_t - a\Delta u = f(u, v, \nabla u, \nabla v),$$
$$v_t - c\Delta u - d\Delta v = g(u, v, \nabla u, \nabla v),$$

with homogeneous Dirichlet or Neumann boundary conditions and initial data u_0, v_0.

Using the linear transformation $\tilde{v} = v - \frac{c}{a-d}u$ (with $a > d > 0$, $c^2 < 4ad$), the system decouples. The proof uses compact semigroups $\{S_a(t), S_d(t)\}$ generated by $a\Delta, d\Delta$ in $L^1(\Omega)$, a priori L^1 and L^2 estimates via truncation and convolution.

2. Introduction

The abstract Cauchy problem for reaction-diffusion systems appears in combustion, chemical reactions, biology, ecology (Britton , Fife [2]).

For special cases, Alikakos proved L^∞ bounds for $f = -uv^\sigma$ with $1 < \sigma < \frac{n+2}{n}$; Masuda extended to all $\sigma \geq 1$.

For $c = 0$, Alaa and Mounir [1] treated the system as a perturbation.

For $c \geq 0$, Kirane [3] obtained global bounds under $v_0 \geq \frac{c}{a-d}u_0$.

Our goal: establish global existence for general nonlinearities with critical gradient growth, and study asymptotic behavior via Lyapunov methods.

3. Mathematical Model

Original system :

$$\begin{cases} u_t - a\Delta u = f(u, v, \nabla u, \nabla v), \\ v_t - c\Delta u - d\Delta v = g(u, v, \nabla u, \nabla v), \end{cases}$$

on $Q_T = (0, T) \times \Omega$, with $\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0$ or $u = v = 0$ on Σ_T, and initial data $u(0) = u_0$, $v(0) = v_0$. Assume $a > d > 0$, $c \geq 0$, $c^2 < 4ad$.

Define the linear transformation (6):

$$\tilde{v} = v - \frac{c}{a-d}u \quad \Longleftrightarrow \quad v = \tilde{v} + \frac{c}{a-d}u.$$

Then the system becomes :

$$\begin{cases} u_t - a\Delta u = \tilde{f}(u, \tilde{v}, \nabla u, \nabla \tilde{v}), \\ \tilde{v}_t - d\Delta \tilde{v} = \tilde{g}(u, \tilde{v}, \nabla u, \nabla \tilde{v}), \end{cases}$$

with the same boundary and initial conditions.

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4. Key Hypotheses

- **Sign condition:** $u\tilde{f}(u, \tilde{v}, \nabla u, \nabla \tilde{v}) \leq 0$ for all $u, \tilde{v} \geq 0$ a.e.
- **Positivity preservation:** $\tilde{f}(0, \tilde{v}, 0, s) \geq 0$, $\tilde{g}(u, 0, r, 0) \geq 0$.
- **Mass control:** $\tilde{f} + \tilde{g} \leq L_1(u + \tilde{v} + 1)$, $L_1 > 0$.
- **Local Lipschitz continuity.**
- **Critical growth:** $|\tilde{f}| \leq C_1(|u|)\big(\hat{F}(t, x) + |\nabla u|^2 + |\nabla \tilde{v}|^\alpha\big)$, $1 \leq \alpha < 2$.
- **Similar condition for** $\tilde{g}$.

5. Semigroup Framework

Theorem 1: $A = \Delta$ with Neumann b.c. is m-dissipative in $L^p(\Omega)$ $(1 < p < \infty)$. Its restriction to $C(\bar{\Omega})$ is also m-dissipative.

Remark 1: The semigroup $S(t)$ generated by Δ is compact in $C(\bar{\Omega})$.

Theorem 2: Let $T > 0$. A function $u : [0, T] \to X$ is a weak solution of (5) if and only if $F(u(t)) \in L^1(0, T, X)$ and u satisfies the variation of constants formula:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s), \nabla u(s))\, ds,$$

for all t$\in [0, T]$.

Theorem 3: If F is locally Lipschitz, there exists a unique maximal solution $u(t)$ on $[0, T_{\max})$ such that either $T_{\max} = \infty$ (global existence) or $\lim_{t \to T_{\max}^-} \|u(t)\| = +\infty$.

6. Main Result

Under the above hypotheses , if $u_0, \tilde{v}_0 \in L^2(\Omega)$, then there exists a positive global weak solution $(u, \tilde{v})$ of the system such that for any $T > 0$:

$$u, \tilde{v} \in C([0, T]; L^1(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega)),$$
$$u, \tilde{v} \in L^2(0, T; H_0^1(\Omega)), \quad \tilde{f}, \tilde{g} \in L^1(Q_T).$$

7. A Priori Estimates

Define truncation: $T_k(s) = \max(-k, \min(k, s))$, $G_k(s) = s - T_k(s)$.

For approximating sequences $(u_n, \tilde{v}_n)$ via convolution with ρ_n:

Lemma 1.

$$\|u_n(t) + \tilde{v}_n(t)\|_{L^1(\Omega)} \leq M.$$

Lemma 2.

$$\int_{Q_T} |\nabla u_n|^2 \leq R_2, \quad \int_{Q_T} |\nabla \tilde{v}_n|^2 \leq R_3.$$

Lemma 3.

$$\int_{Q_T} (2u_n + \tilde{v}_n)(|\tilde{f}_n| + |\tilde{g}_n|) \leq R_4.$$

These estimates yield convergence via Arzelà–Ascoli and compactness.

8. Asymptotic Behavior

Treating the system as a dynamical system in $C(\bar{\Omega}) \times C(\bar{\Omega})$ and using Lyapunov-type techniques, precompactness of orbits (uniform C^1 bounds for $t > 0$) together with compactness of the semigroup implies convergence to steady states as $t \to +\infty$. For $c = 0$ and nonnegative data, or for $c \geq 0$ with $v_0 \geq \frac{c}{a-d}u_0$, the solution converges to a constant vector (k_1, k_2) satisfying $k_1\Psi(k_2) = 0$ (Kirane [3]).

9. Conclusion

We have established the global existence of unique weak solutions for a class of abstract Cauchy problems using compact semigroup methods and L^1 estimates.

Moreover, the asymptotic analysis shows convergence to steady states under appropriate conditions.

Future work will focus on obtaining explicit decay rates, asymptotic profiles, and numerical simulations to illustrate the theoretical results.

References

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[3] M. Kirane, *Global bounds and asymptotics for a system of reaction-diffusion equations*, J. Math. Anal. Appl. 138 (1989), 328–342.