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Fractional-Time Jaynes-Cummings Model with Unitary Description: Dynamics for Initial Binomial State of Light

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INTRODUCTION & AIM

Fractional calculus is a branch of mathematics that generalizes the differentiation operator to real and complex orders. In the context of quantum mechanics, the two main generalizations are the fractional-space Schrödinger equation and the fractional-time Schrödinger equation (FTSE), the latter of which results in non-unitary evolution.

In this work, we analyse the quantum dynamics of the fractional-time Jaynes-Cummings (JC) model with an initial binomial state of light. To address the non-unitarity, we use a recent unitary framework for the FTSE. We examine how the fractional derivative order α influences non-classical features under different initial conditions.

METHOD

The JC model depicts the interaction between a quantized cavity mode and a two-level atom. Under resonance, the JC model Hamiltonian in the interaction picture is given by ($\hbar = 1$)

$$\hat{H} = \mu(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger).$$

The FTSE is given by

$$i^\alpha\, {}_0^{\mathrm{C}}\mathcal{D}_t^\alpha \left|\Psi_\alpha(t)\right\rangle = \hat{H}_\alpha \left|\Psi_\alpha(t)\right\rangle,$$

where the Caputo derivative is given by

$${}_0^{\mathrm{C}}\mathcal{D}_t^\alpha(\cdot) = \frac{1}{\Gamma(1-\alpha)} \int_0^t d\tau (t-\tau)^{-\alpha} \frac{d}{d\tau}(\cdot),$$

with $\alpha \in [0,1)$.

The non-unitary evolution operator that results from the FTSE in the JC model is

$$\hat{U}_\alpha^{(n)}(t) = \begin{pmatrix} \mathcal{C}_\alpha^{(n)}(t) & i^{-\alpha}\mathcal{S}_\alpha^{(n)}(t) \\ i^{-\alpha}\mathcal{S}_\alpha^{(n)}(t) & \mathcal{C}_\alpha^{(n)}(t) \end{pmatrix},$$

with the auxiliary functions ($E_\alpha(x)$ being the Mittag-Leffler function):

$$\mathcal{C}_\alpha^{(n)}(t) = \frac{E_\alpha(i^{-\alpha}\mu_\alpha^{(n)}t^\alpha) + E_\alpha(-i^{-\alpha}\mu_\alpha^{(n)}t^\alpha)}{2},$$

$$\mathcal{S}_\alpha^{(n)}(t) = \frac{E_\alpha(i^{-\alpha}\mu_\alpha^{(n)}t^\alpha) - E_\alpha(-i^{-\alpha}\mu_\alpha^{(n)}t^\alpha)}{2i^{-\alpha}}.$$

The unitary evolution operator is obtained with time-dependent Dyson maps, via the similarity transformation:

$$\hat{u}_\alpha(t) = \hat{\eta}_\alpha(t)\,\hat{U}_\alpha(t)\,\hat{\eta}_\alpha^{-1}(0).$$

The chosen map for the problem is

$$\hat{\eta}_\alpha^{(n)}(t) = \frac{e^{\kappa_\alpha^{(n)}(t)}}{\sqrt{\Lambda_\alpha^{(n)}(t)}} \begin{pmatrix} \chi_\alpha^{(n)}(t) & \lambda_\alpha^{(n)}(t) \\ [\lambda_\alpha^{(n)}(t)]^* & 1 \end{pmatrix}.$$

For our computations, we consider that the atom is initially excited and the cavity mode is in the binomial state

$$|B_p^M;\theta_n\rangle = \sum_{n=0}^{M} e^{i\theta_n} B_{n,p}^M |n\rangle,$$

where we will set $\theta_n = 0$ and

$$B_{n,p}^M = \left[\frac{M!}{n!(M-n)!} p^n (1-p)^{M-n}\right]^{1/2}.$$

The unitarily time-evolved state then takes the general form

$$|\psi_\alpha^p(t)\rangle = \sum_{n=0}^{\infty} \left[A_{e,n}^{\alpha,p}(t)|e,n\rangle + A_{g,n}^{\alpha,p}(t)|g,n+1\rangle\right].$$

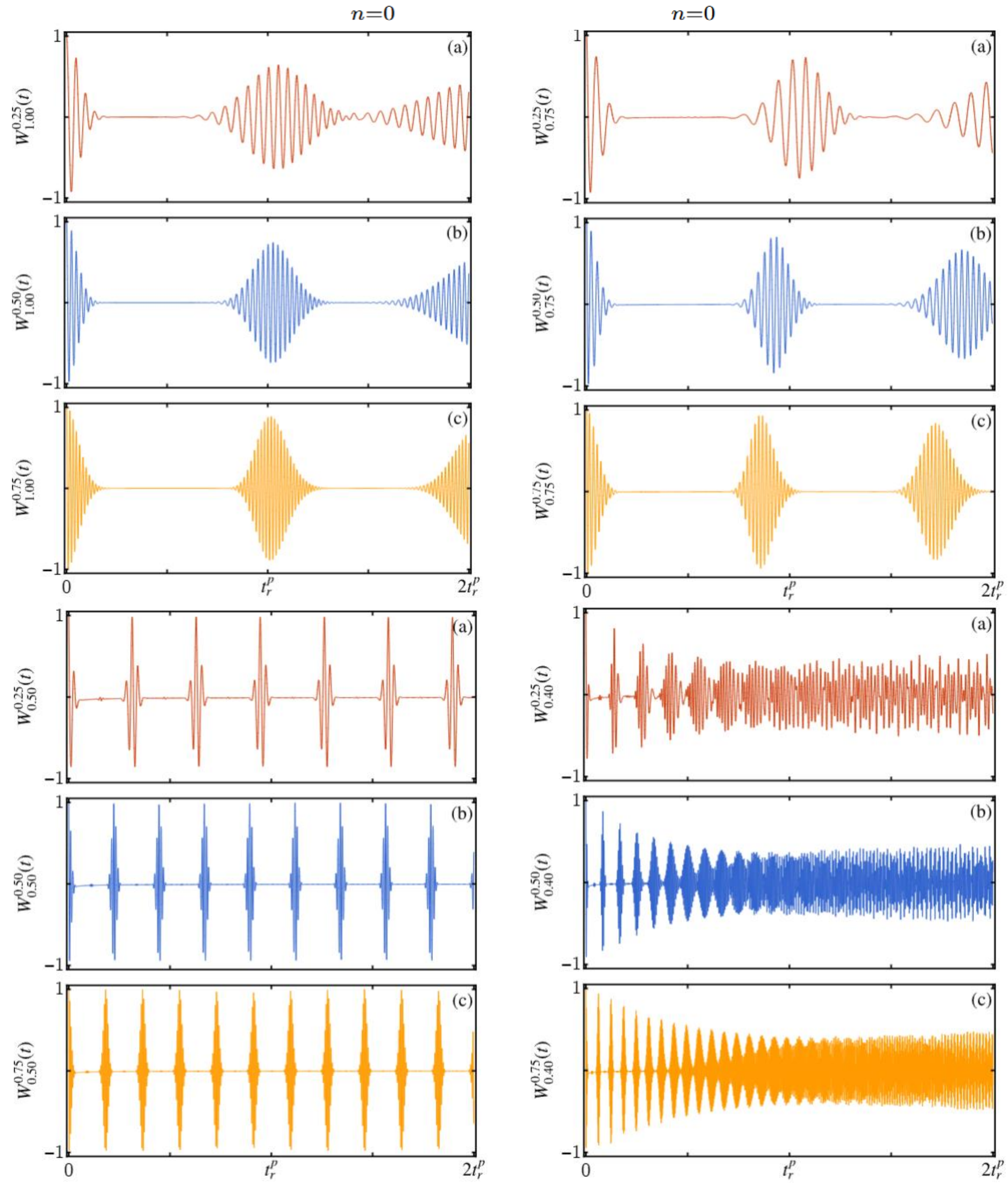
RESULTS & DISCUSSION

As a figure of merit, we focus on the population inversion

$$W_\alpha^p(t) = P_e^{\alpha,p}(t) - P_g^{\alpha,p}(t),$$

where

$$P_e^{\alpha,p}(t) = \sum_{n=0}^{\infty} |A_{e,n}^{\alpha,p}(t)|^2,\ P_g^{\alpha,p}(t) = \sum_{n=0}^{\infty} |A_{g,n}^{\alpha,p}(t)|^2.$$



CONCLUSION

In this work, we have analysed the dynamics of an initial binomial state of light within the unitary formulation of the fractional-time JC model. Different derivative orders (α) distinctly influence the dynamics: a decreasing number of oscillations ($\alpha = 0.75$), periodicity ($\alpha = 50$) and aperiodicity ($\alpha = 0.40$).

FUTURE WORK / REFERENCES

[1] R. Herrmann, Fractional Calculus (World Scientific, 2014).

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