IOCMA 2026 Conference
The 2nd International Online Conference on Mathematics and Applications
10–12 June 2026 | Online
MDPI

A Sixth-Order Averaging Theory for Limit Cycles Bifurcating From Uniform Isochronous Cubic Centers

Aya Zaiem , Amar Makhlouf
Department of Mathematics, University of Annaba, Laboratory LMA PO. BOX.12;23 ANNABA, ALGERIA.
Department of Mathematics, University of Annaba, Laboratory LMA PO. BOX.12;23 ANNABA, ALGERIA.

INTRODUCTION & AIM

Limit cycle bifurcation from isochronous centers is a fundamental problem in the qualitative theory of differential systems. In this work, we study the perturbed uniform isochronous cubic center

$$\begin{cases} \dot{x} = -y + x^2y + y^2x + \sum_{i=1}^{6} \varepsilon^i Q_i(x,y) \\ \dot{y} = x + y^2x + x^2y + \sum_{i=1}^{6} \varepsilon^i Q_i(x,y) \end{cases}$$

where $P_i$and $Q_i$are cubic polynomials. Using sixth-order averaging theory, we determine the maximum number of small-amplitude limit cycles bifurcating from the period annulus. Our results show that up to three limit cycles can emerge under small perturbations.

METHOD

The sixth-order averaging method is applied to study limit cycle bifurcations. The positive simple zeros of the sixth averaged function determine the emerging limit cycles.

1. Perturbed System
Consider the uniform isochronous cubic center under polynomial perturbations up to sixth order in ε.
2. Coordinate Transformation
Transform the system into a standard radial equation using suitable polar-type coordinates.
3. Sixth-Order Averaging
Compute the averaged functions $f_1, \dots, f_6$recursively and impose

$$f_1 = f_2 = f_3 = f_4 = f_5 \equiv 0.$$

4. Bifurcation Equation
Obtain the sixth averaged function

$$f_6(r) = r\,P(r^2),$$

where the coefficients of Pdepend on the perturbation parameters.
5. Limit Cycles Detection
The positive simple zeros of $f_6$correspond to limit cycles. The number of such zeros gives the maximum number of small-amplitude limit cycles.

RESULTS & DISCUSSION

The sixth-order averaging method was applied to the perturbed uniform isochronous cubic center. After imposing the conditions

$$f_1(r) = f_2(r) = f_3(r) = f_4(r) = f_5(r) = 0.$$

the bifurcation problem is reduced to the analysis of the sixth averaged function

$$f_6(r) = r(A_5r^4 + A_4r^3 + A_3r^2 + A_1).$$

The coefficients A_1, A_3 , A_4 and A_5 can be independently controlled through suitable perturbation parameters. This independence allows the application of algebraic techniques to determine the number of positive zeros of $f_6(r)$.
Since each positive simple zero of $f_6(r)$corresponds to a limit cycle of the perturbed system, we conclude that at most three small- amplitude limit cycles bifurcate from the period annulus of the uniform isochronous cubic center.
This result is obtained through sixth-order averaging and extends previous results based on lower-order averaging methods.

CONCLUSION

Sixth-order averaging theory was applied to perturbed uniform isochronous cubic centers. The obtained bifurcation equations allow the detection of small-amplitude limit cycles and show the effectiveness of higher-order averaging in revealing complex dynamics. This approach can be extended to other classes of polynomial differential systems with isochronous centers.

FUTURE WORK / REFERENCES

J. Sanders, F. Verhulst, J. Murdock, Averaging Methods in Nonlinear Dynamical Systems, Springer.
A. Buică, J. Llibre, Averaging methods for finding periodic orbits.
V. Pleshkan, Classification of cubic isochronous centers.
C. Christopher, C. Li, Limit Cycles of Differential Equations.