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Investigation of Third-Order Refinement Iterative Methods for Solving Linear Systems in Engineering Applications

K. J. Audu[1]*, R. A. Oyewole[1], Y. A.Yahaya[1], M. F. Inuolaji[1] and F. A. Olayiwola[1]

[1]Department of Mathematics, Federal University of Technology, Minna, Nigeria. **Email**: k.james@futminna.edu.ng

INTRODUCTION & AIM

Most Engineering problems such as structural analysis heat-transfer, electrical circuit design and fluid mechanics [3,4] gives linear systems which can be represented as:

$$Tx = u \quad (1)$$

where $T \in {}^{nxn}$ is the coefficient matrix, x is the unknown vector and $u \in {}^{n}$ is the right-hand side vector. The coefficient matrix T can be splitted as

$$T = W - V - Y \quad (2)$$

where W is the diagonal of T, -V and –Y are the strictly upper and lower triangular matrices of T.

Statement of the problem

Traditional iterative method fails in achieving the desired level of accuracy for a large-scale Engineering based linear system. Hence this research work applies the TRG and TRGS to explore their effectiveness in improving convergence rate.

METHOD

Taking (1) into consideration, the matrix form of Jacobi method is derived by joining (1) and (2) to give;

$$x^{(s+1)} = W^{-1}(V+Y)x^{(s)} + W^{-1}u \quad (3)$$

Then, the Refinement of Jacobi(RJ) method is expressed as

$$x^{(s+1)} = \left[W^{-1}(V+Y)\right]^2 x^{(s)} + \left[I + W^{-1}(V+Y)\right]W^{-1}u \quad (4)$$

Further adjustment of (4) gives the TRJ [2] expressed as

$$x^{(s+1)} = \left[W^{-1}(V+Y)\right]^4 x^{(s)} + \Big[I + W^{-1}(V+Y) + \left(W^{-1}(V+Y)\right)^2 + \left(W^{-1}(V+Y)\right)^3\Big]W^{-1}u \quad (5)$$

The matrix form of Gauss-Siedel method in (6) is derived by combining (1) and (2)

$$x^{(s+1)} = \left[(W-\bar{V})^{-1}Y\right]x^{(s)} + (W-V)^{-1}u \quad (6)$$

The Refinement of Gauss Siedel is obtained as

$$x^{(s+1)} = \left[(W-V)^{-1}Y\right]^2 x^{(s)} + \left[I + (W-V)^{-1}Y\right](W-V)^{-1}u \quad (7)$$

Algebraic manipulations of (7) gives the TRGS [1] as

$$x^{(s+1)} = \left[(W-V)^{-1}Y\right]^4 x^{(s)} + \Big[I + (W-V)^{-1}Y + \left((W-V)^{-1}Y\right)^2 + \left((W-V)^{-1}Y\right)^3\Big](W-V)^{-1}u \quad (8)$$

Equation (5) and (8) will be used to solve Engineering-based LS.

RESULTS & DISCUSSIONS

Exp 1: Electrical Circuit Flow (500 x 500 System)

Table 1: Computational Results for the Nodal Voltage Analysis (n = 500,) $\varepsilon=10^{-8}$

Methods	Matrix Size	Iteration Counts	Convergence Rate	CPU Time(s)	Spectral Radius
Std. Jacobi	500 x 500	143	0.891847	0.0394	0.89132917
Std. GS	500 x 500	79	0.804768	0.3383	0.79446770
TRJ	500 x 500	52	0.709393	0.0254	0.70813223
TRGS	500 x 500	29	0.521244	0.2233	0.50145126

Table 1 shows the effectiveness of TRJ and TRGS by converging to the desired solution much faster than their standard methods in the 500 x 500 Engineering- based LS.

Exp 2: A bridge Ladder Resistor (300 x 300 System)

Table 2: Computational Outputs for the Mesh Current Analysis System (n = 300,) $\varepsilon=10^{-77}$

Methods	Matrix Size	Iteration Counts	Convergence Rate	CPU Time (s)	Spectral Radius
Std. Jacobi	300 x 300	141	0.821029	0.0039	0.824603
Std. GS	300 x 300	81	0.700790	0.1038	0.703200
TRJ	300 x 300	38	0.455483	0.0062	0.560705
TRGS	300 x 300	22	0.242013	0.1163	0.347725

This Table shows how TRJ and TRGS outperform the Std. Jacobi and Std. GS by taking fewer convergence rate and iteration number. Despite their effectiveness, TRGS shows the best performance in the 300 x 300 Engineering- based LS.

Exp 3: Structural Engineering (400 x 400 System)

Table 3: Computational Results for the Planar Truss system (n = 400,) $\varepsilon=10^{-}$[illegible]

Methods	Matrix Size	Iteration Counts	Convergence Rate	CPU Time (s)	Spectral Radius
Std. Jacobi	400 x 400	179	0.878754	0.0254	0.890835
Std. GS	400 x 400	96	0.781805	0.3345	0.794038
TRJ	400 x 400	71	0.712860	0.0117	0.706956
TRGS	400 x 400	32	0.502435	0.3764	0.500638

The Table shows the performance of TRJ and TRGS in achieving great in reducing iteration counts, CPU Time, and Spectral Radius for the 400 x 400 Engineering- based LS compared to their Standard methods.

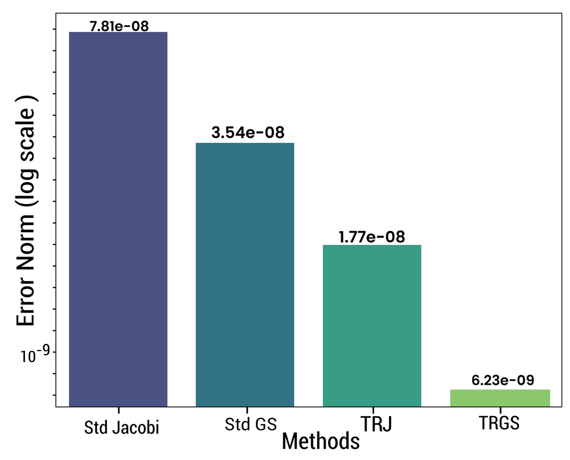

Figure 1: Error Norm Plot for Exp 1

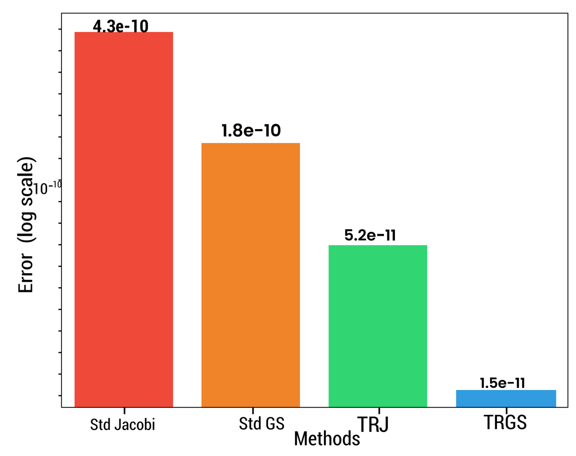

Figure 2: Error Norm Plot for Exp 2

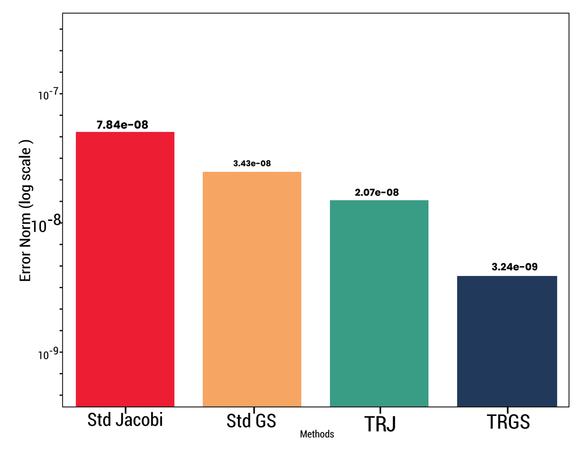

Figure 3: Error Norm Plot for Exp 3

- Figure 1 shows that, across all four methods, TRGS achieved the lowest error norm and the highest solution accuracy.
- Figure 2 depicts that TRGS outperforms all other methods, demonstrating its superior convergence properties.
- Figure 3 illustrates that TRGS maintained the lowest error values throughout the process, making it the most accurate method when compared to the others..

CONCLUSIONS

- TRGS outperforms TRJ in a range of engineering benchmarks and refinement techniques.
- The smaller the spectral radius, the faster the convergence, which implies a lower number of iterations.
- Third refinement methods provide reliable numerical solutions for linear systems that are often used in engineering.
- No extra matrix factorization is required for the third refinement methods.

FUTURE WORK/REFERENCES

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