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Entropy as a metric generator of dissipation in complete metriplectic systems

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Abstract: This lecture is a short review on the role entropy plays in classical dissipative dynamics formulated in terms of *Leibniz bracket algebræ* (LBA). While conservative dynamics is given an LBA formulation in the Hamiltonian framework, with total energy H generating the motion via classical Poisson brackets or quantum commutation brackets, an LBA formulation can be given to classical dissipative dynamics through the *metriplectic bracket algebra* (MBA): the conservative component of the dynamics is still generated via a Poisson algebra by the total energy H, while S, the entropy of the degrees of freedom statistically encoded in friction, generates dissipation via a metric bracket. Here a (necessarily partial) overview on the types of systems subject to MBA formulation is presented, and the physical meaning of the quantity S involved in each is discussed. At the end of this collection of examples, the fact that dissipative dynamics may be constructed also in the absence of friction with microscopic degrees of freedom is stressed. This reasoning is a hint to introduce dissipation at a more fundamental level.

Keywords: dissipative systems; metriplectic dynamics; Leibniz algebræ

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1. Introduction

When the Organizers of the *2nd International Electronic Conference on Entropy and Its Applications* invited me to submit a keynote lecture in the section of Physics, I was not only honoured and delighted for their kindness, but also impressed by the wide choice of things about "entropy" one

could discuss. My choice has been to report on the role that this still mysterious object, *the entropy*, seems to play in the context of classical dissipative dynamics represented through bracket algebræ.

Entropy, indeed, is known to play the role of a thermodynamic state function, to measure the degradation of the energy transferred, as well as the spread of it through microscopic degrees of freedom; close or far relatives of entorpy are employed to quantify ignorance or unpredictability, and even to track causality in the complex relationships of interacting systems. In the context of algebrized dynamics of classical dissipative systems, entropy plays the same role of energy, when the latter wears the costume of the Hamiltonian H: entropy contributes to generating the motion of the system, even if through a mechanism different from the Hamiltonian one. According to *metriplectic formalism*, the subject of this lecture, the evolution of a classical system with dissipation is a 1-parameter (*time*) semi-group resulting from the exponentiation of the sum of the *symplectic bracket* $\{...,H\}$ ruled by H, plus a *metric bracket* $(...,S)$ ruled by entropy S.

$$Motion = \exp(\{...,H\}+(...,S))$$

In this way entropy enters directly and vividly in the shoes of *εν τροπη*, a Greek phrase translatable as "the inner transformer", to which the physical function owes its name. Impressively, while Hamiltonian and symplectic machinery just produce *time-reversible* changes, what entropy does with metric algebra is generating *irreversible* transformations. To the opinion of the Author of this lecture, this fact is extremely intriguing and possibly related to the long-standing question of the origin of irreversibility in Physics. I just wrote "possibly related to", that does mean "resolving": however, I think that students or researchers aiming at contributing to the work about the just mentioned dilemma should have some knowledge of metriplectic formalism.

In this lecture I will discuss about dissipative *complete systems*, i.e. pieces of the universe that conserve their energy while increasing their entropy during their evolution, i.e. essentially isolated systems: so, I apologise in advance with all those who work on the metriplectic formalism applied to non-isolated (e.g., non energy-conserving) systems: those systems are extremely important examples that, however, I had not to mention here, to keep this lecture not uselessly long.

The presentation is articulated as follows.

In Section 2 the concept of dynamics algebrization through Leibniz bracket algebræ is reported, and the usefulness of representing dynamics via bracket algebræ highlighted. Hamiltonian systems and metric dissipative systems are recognized among these dynamical theories.

Section 3 is the core of the keynote lecture: the concept of complete metriplectic system is described, and the relationship between dissipation and friction is discussed. In particular, it is stressed how it is possible to construct dissipative dynamics even in the absence of "explicit friction" via metriplectic algebra. The student will find § 2 rich of examples, taken for relatively elementary Physics, to get acquainted with metriplectic formalism applied to isolated systems.

Section 4 is finally devoted to some final considerations on the potentiality and meaning of metriplectic formalism.

2. Leibniz Bracket Algebra and Dynamics

A Leibniz bracket algebra is defined by two elements: a manifold M, on which the space of (real) smooth functions $C^\infty(M,\mathbf{R})$ is defined, and a $C^\infty(M,\mathbf{R})\times C^\infty(M,\mathbf{R})\mapsto C^\infty(M,\mathbf{R})$ map, namely *the bracket* $(.,.)_\mathrm{L}$, that associates to two functions f and g in $C^\infty(M,\mathbf{R})$ a third smooth function

$$\ell = (f,g)_\mathrm{L}. \tag{1}$$

The bracket must have the following properties [1]:

$$\begin{cases} \left(\sum_i \lambda_i f_i, \sum_j \mu_j g_j\right)_\mathrm{L} = \sum_{i,j} \lambda_i \mu_j (f_i, g_j)_\mathrm{L}, \\ (f_1 f_2, g)_\mathrm{L} = f_1 (f_2, g)_\mathrm{L} + f_2 (f_1, g)_\mathrm{L}, \\ (f, g_1 g_2)_\mathrm{L} = g_1 (f, g_2)_\mathrm{L} + g_2 (f, g_1)_\mathrm{L}, \end{cases} \tag{2}$$

where λ_i and μ_j are arbitrary real coefficients. The properties (2) make the Leibniz bracket *derivative in both its arguments*.

If one of the two functions f and g defining ℓ in (1) is fixed, let us indicate it as F, then the application $(.,F)_\mathrm{L}$ maps $C^\infty(M,\mathbf{R})$ onto vector fields on M: indeed the map

$$X_F f \overset{\text{def}}{=} (f,F)_\mathrm{L} \tag{3}$$

does have all the properties of a vector. The vector field X_F in (3) defines a flux throughout M, and then a dynamics of any variable: this observation is the essence of Leibniz dynamics, or dynamics on Leibniz algebræ. The function $F \in C^\infty(M,\mathbf{R})$ is referred to as *Leibniz generator of the dynamics* given by X_F. In a more physical way, defining a scalar evolution parameter t, it is possible to assign the dynamics of functions $f \in C^\infty(M,\mathbf{R})$ by stating:

$$\dot{f} = (f,F)_\mathrm{L}, \tag{4}$$

being $\dot{f} = \frac{df}{dt}$. Provided M is the phase space of a physical system, the ODE (4) is the equation of motion of any observable as it descends from the motion of the system configuration through it. Before moving on, it is important to stress that *a dynamics expressed as in (4) reinterprets the evolution as a transformation of functions on M*, since infinitesimal variations of an observable is understood as $\delta f = (f,F)_\mathrm{L}\delta t$: this is the exquisite idea of *algebrizing dynamics*, i.e. reducing the evolution (with time) to a set of algebraic transformations [3]. Next to this, there comes the taxonomy of algebraic structures describing "chains of transformations", in particular *groups and semigroups* [2].

In order to have a more concrete taste of what Leibniz dynamics is, consider M as described by a (global) chart of real coordinates, so that a (not necessarily finite…) collection of dynamical variables $\mathbf{x}$ describes each position in M; hence, if the symbol $\partial_i = \frac{\partial}{\partial x^i}$ is used to indicate the derivative with respect to the i-th component of $\mathbf{x}$, the Leibniz bracket $(f,g)_\mathrm{L}$ can be calculated as

$$(f(\mathbf{x}),g(\mathbf{x}))_{\mathrm{L}} = L^{ij}(\mathbf{x})\partial_i f(\mathbf{x})\partial_j g(\mathbf{x}), \tag{5}$$

being $L^{ij}(\mathbf{x})$ a smooth rank-2 tensor mapping M into local $\dim M \times \dim M$ matrices. A Leibniz bracket is hence assigned as the tensor field $L^{ij}(\mathbf{x})$ is given, whereas in order for a Leibniz dynamics to be assigned one also needs a dynamical generator $F(\mathbf{x})$ as in (4). Note that in (5) the repeated indices in contravariant position are summed on, as throughout the whole lecture.

Some advantages appear in formulating the dynamics of a system in terms of Leibniz brackets as in (4). First of all, one has *covariance of equations under diffeomorphisms of M*. Indeed, considering (5), and provided the evolution parameter t does not depend on the coordinate system on M, if the observables f and g are "scalar" functions, the expression

$$\dot{f} = L^{ij}\partial_i f \partial_j F \tag{6}$$

is invariant under changes of coordinates in M. This means that if the Physics of a system can be reformulated via a Leibniz algebra its equations of motion have a form independent of the choice of how to describe the phase space. Moreover, if instead of f one inserts the k-th component of $\mathbf{x}$ in (6), the relationship

$$\dot{x}^k = L^{kj}\partial_j F \tag{7}$$

is obtained, describing just "the motion of the system" through its phase space M. This expression indicates that the velocity of the system across M, of components $\dot{x}^k$, transforms under diffeomorphisms as the 1-form dx^k, i.e. it is a proper vector.

The other important advantage in expressing dynamics through a bracket algebra is that the *symmetries of the dynamics* become apparent. Indeed, from equation (6), any physical quantity f the gradient of which is a null mode of L is conserved throughout the motion:

$$L^{kj}\partial_k f = 0 \quad \Rightarrow \quad \dot{f} = 0. \tag{8}$$

If so, then f has null bracket with any other observable:

$$L^{kj}\partial_k f = 0 \quad \Rightarrow \quad (f,g)_{\mathrm{L}} = 0 \quad \forall \quad g \in C^{\infty}(M,\mathbf{R}). \tag{9}$$

Of course, in order for f to be constant, it is enough that it has null bracket with the dynamics generating observable F:

$$(f,F)_{\mathrm{L}} = 0 \quad \Rightarrow \quad \dot{f} = 0. \tag{10}$$

In general, *all the observables with zero bracket with F are constant*.

About the generating function F, its mathematical aspect determines *the possible steady states* of the system: indeed, thanks to (6) and (7), one may state

$$\partial_j F(\mathbf{x}_0) = 0 \quad \Rightarrow \quad \dot{x}^k(\mathbf{x}_0) = 0, \quad \dot{f}(\mathbf{x}_0) = 0. \tag{11}$$

If the system is abandoned with null velocity precisely at a point $\mathbf{x}_0$ where the gradient of F vanishes, then it does not move away from there (of course, the stability of such a stationary point is another matter). Note that equation (11) does not exclude that other points are stationary configurations of the system in M: instead, points with null gradient of F represent the steady points in M of the system, if the tensor L^{ij} is non singular.

The generator F undergoes the same Leibniz dynamics as in (6): the relationship

$$\dot{F} = L^{ij}\partial_i F \partial_j F$$

may yield conservation, decrease or increase of F, depending on the algebraic local characteristics of the tensor L: where the tensor is semidefinite positive or negative, then F will locally increase or decrease, being $\dot{F} \geq 0$ or $\dot{F} \leq 0$ respectively.

2.1. Hamiltonian Systems and Poisson Algebræ

Fundamental Physics is expressed in terms of a Leibniz algebra: in particular, as the Classical Analytical Mechanics and then Quantum Mechanics teach, the microscopic fundamental systems in Physics are considered Hamiltonian systems, for which a Hamiltonian observable H is written, and the dynamics is generated through *Poisson brackets*. Indeed, Poisson brackets are an example of Leibniz algebra [5].

Poisson brackets, that are indicated here as $\{.,.\}$, show all the properties (2); moreover, they are anti-symmetric and satisfy the Jacobi identity:

$$\begin{cases} \{f,g\} = -\{g,f\} \quad \forall \quad f,g \in C^{\infty}(M,\mathbf{R}), \\ \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0. \end{cases} \tag{12}$$

When the equation (5) is written for Poisson brackets

$$\{f(\mathbf{x}),g(\mathbf{x})\} = J^{ij}(\mathbf{x})\partial_i f(\mathbf{x})\partial_j g(\mathbf{x}) \tag{13}$$

one then assigns suitable properties to the tensor J so that the relationships (12) are satisfied. In particular, its anti-symmetric nature $J^{ij} = -J^{ji}$ and a differential relationship corresponding to the Jacobi identity $J^{ih}\partial_h J^{jk} + J^{kh}\partial_h J^{ij} + J^{jh}\partial_h J^{ki} = 0$ are required for J to define a Poisson bracket [4].

The dynamics generator for Hamiltonian systems is indicated as H, the *Hamiltonian*. From the relationship

$$\dot{f} = \{f,H\},$$

reading $\dot{x}^i = J^{ij}\partial_j H$ for the dynamical variables describing the state of the system, and due to the anti-symmetric property of $\{.,.\}$, one has

$$\dot{H} = 0. \tag{14}$$

Equation (14) is simply the energy conservation: the dynamics generator of a Hamiltonian system is conserved throughout the motion. Examples of a Hamiltonian system are not provided here, but resumed in § 3 (besides being very abundant in literature).

Observables in Hamiltonian systems fulfilling the condition (8) and (9), that are written here as

$$J^{kj}\partial_k f = 0 \quad \Rightarrow \quad \{f,g\} = 0 \quad \forall \quad g \in C^{\infty}(M,\mathbf{R}),$$

are referred to as *Casimir observables* [4]: this name unveils a relationship, indeed, between Poisson bracket algebræ and the world of *Lie groups*, where a Casimir is a fundamental algebraic invariant of the group algebra, depending on the algebra elements themselves. In the context of Hamiltonian systems the Leibniz brackets at hand satisfy (12), then they show the same structure as a Lie algebra, hence the name of functions having null Poisson bracket with any other observable. By the way, for

suitable expressions of the tensor J^{ij} the expression (13) may reproduce the generating algebra of an otherwise known Lie group (in this case one speaks about *Poisson-Lie algebra*).

Due to the properties of Poisson brackets, the Hamiltonian systems trajectories in M cannot converge to an attracting point, or become confined to an attractor; instead, trajectories of Hamiltonian systems are either unbounded, or they "eternally" turn, regularly or not, without ever "stopping" at a point. In particular, one may state that *Hamiltonian systems do not admit asymptotically stable equilibria*. This is why Hamiltonian dynamics is perfect to describe the evolution of "immutable" systems: they may just redistribute energy among their degrees of freedom *in a reversible way*, as in the case of an ideal pendulum that continuously turns its gravitational energy into kinetic one and then backwards, but cannot "bury" or "forget" energy in a definite form forever. This means that their trajectories throughout M could be physically swept backward while the system remains perfectly isolated.

2.2. Dissipative Systems and Metric Algebræ

Now, suppose the tensor L in (6) to be a *semimetric tensor* G, i.e. a symmetric, semi-definite (e.g., positive) tensor:

$$(f,g) = G^{ij}\partial_i f \partial_j g \quad / \quad G^{ij} = G^{ji}, \quad G^{ij}\partial_i f \partial_j f \geq 0 \quad \forall \quad f,g \in C^{\infty}(M,\mathbf{R}). \tag{15}$$

No Jacobi identity is satisfied by such a bracket. The bracket $(.,.)$, referred to as *(semi)metric bracket*, is a type of Leibniz bracket quite different from Poisson algebræ [6].

Once a generating function $Q(\mathbf{x})$ is defined so that the dynamics of the system is governed by

$$\dot{x}^i = G^{ij}\partial_j Q, \quad \dot{f} = (f,Q), \tag{16}$$

it is possible to make some statements about this particular kind of system.

The essential fact to pick up in this discussion is that the function generating the dynamics, i.e. the Q in (16), is not constant; in particular, if G is a semi-definite positive tensor, then Q tends to grow monotonically along the evolution

$$\dot{Q} = (Q,Q) \geq 0: \tag{17}$$

it is possible to show that *isolated maxima of Q are asymptotically stable equilibrium points*. This is readily proved (see [6]) realizing that, thanks to (17), Q is a *Lyapunov quantity*, so that points $\mathbf{x}_0$ such that $\partial_i Q(\mathbf{x}_0) = 0$ are places towards which the motion (16) will converge. If G were a semi-definite negative tensor, equally some *Q'* could be defined to play the opposite role, i.e. that of a monotonically decreasing quantity, and the asymptotically stable equilibrium points would then be its minima.

It is very important to have asymptotically stable equilibria into which trajectories converge, because then the system may be used to represent a *dissipative process*: dissipation drives systems in different states of motion to converge to a steady state. This steady state is characterized by either a minimum of non-thermal energy (for open systems the energy of which is drained by friction) or a maximum of entropy (for closed "complete" systems obtained including the degrees of freedom responsible for dissipation, see § 3): in both cases *one may use the quantity monotonic in time to define a Leibniz dynamics representing the system*. As a consequence, metric dynamics is perfect to mimic systems *evolving in an irreversible way*, for which the energy is *transferred from a form into another and cannot come back by virtue of the same equations of motion*. In other words, *metric systems evolve*

irreversibly and age (their only feasible history is that making Q grow, or decrease, forever, according to the sign of $\det G$). While Hamiltonian systems have the conservation of energy as their pivoting principle, metric systems have the *increase of entropy* as their guiding law. Of course, the most complicated question is what interpretation one should give to the concept of "entropy".

3. Complete Metriplectic Systems

In § 2.1 Leibniz systems have been described with the tensor $L = J$ in (8) anti-symmetric, and satisfying the Jacobi identity; in § 2.2, instead, the case of Leibniz dynamical systems with tensor $L = G$ symmetric, and semi-definite, is discussed. In general, the structure described in § 2 has an L that is neither symmetric nor antisymmetric, and is composed by both these two parts:

$$L = J + G \quad / \quad J^{ab} = -J^{ba}, \quad G^{ab} = G^{ba}.$$

If the anti-symmetric part J is a Poisson tensor satisfying the relationship

$$J^{ih}\partial_h J^{jk} + J^{kh}\partial_h J^{ij} + J^{jh}\partial_h J^{ki} = 0,$$

namely the Jacobi identity, and if G is semi-definite (for instance, positive semi-definite)

$$G^{ij}\partial_i f \partial_j f \geq 0 \quad \forall \quad f \in C^\infty(M, \mathbf{R}), \tag{18}$$

then the Leibniz system with bracket

$$\langle\langle f(\mathbf{x}), g(\mathbf{x})\rangle\rangle = \left(J^{ij}(\mathbf{x}) + G^{ij}(\mathbf{x})\right)\partial_i f(\mathbf{x})\partial_j g(\mathbf{x}), \tag{19}$$

is referred to as *metriplectic system*, reminding that this is composed of a symplectic and a metric part.

A metriplectic bracket algebra (MBA) can be turned into *a dynamical system* once a *generating function* $F \in C^\infty(M, \mathbf{R})$ is adopted, and the prescriptions

$$\dot{x}^i = \langle\langle x^i, F\rangle\rangle, \quad \dot{f} = \langle\langle f, F\rangle\rangle \quad \forall \quad f \in C^\infty(M, \mathbf{R}) \tag{20}$$

are made. If the definition (19) is used, then the expression for $\dot{f}$ reads

$$\dot{f} = J^{ij}\partial_i f \partial_j F + G^{ij}\partial_i f \partial_j F : \tag{21}$$

in particular, the behavior along the motion of the generating function of a metriplectic system is entirely provided by the character of G, as $J^{ij}\partial_i F \partial_j F = 0$ identically due to the anti-symmetric property of J. One then has

$$\dot{F} = G^{ij}\partial_i F \partial_j F,$$

so that, under the assumption (18), this will be monotonically increasing: $\dot{F} \geq 0$. The generating function F that "evolves the system" through the algebra $\langle\langle ., F\rangle\rangle$ in (21) is hence a Lyapunov quantity of the theory itself, as described in § 2.2: in general, it will contain a part corresponding to the symplectic component of $\langle\langle .,.\rangle\rangle$ in (19), and a part referring to metric algebra. Physical reasoning drives the definition of this and that in § 3.1 below.

3.1. Energy Conservation, Entropy Increase

In [6] a *complete system* is referred to as a system that conserves its energy, but redistributes it in an irreversible way: this "irreversible redistribution" is named *dissipation*. Complete systems described via a MBA are indicated as *complete metriplectic systems* (CMS).

In everyday life, dissipation takes place due to the interaction of "macroscopic" degrees of freedom with "microscopic" ones, and this interaction mode is also named *friction*. When friction is at work, *mechanical or electromagnetic energy is dissipated, i.e. irreversibly transformed, into kinetic energy of the microscopic constituents of the system*, the degrees of freedom of which are, however, *included in the system*. These degrees of freedom are referred to as *Microscopic Statistically Treated Degrees of Freedom* (μSTDoF) [Mate.Tassi.1]; here, "statistically treated" means that what describes these degrees of freedom in the phase space of the whole system is some collective quantities referring to them, precisely their *thermodynamic coordinates*.

3.1.1. In case of friction…

As we'll see from § 3.5 on, friction is not the only way to have dissipation. However, when friction is the pathway to dissipation, the "standard" way to construct a complete system is to consider first a Hamiltonian system, with its energy H_0 and dynamical variables $\mathbf{y}$, that does not undergo any kind of dissipation; then, add dissipation making the system interact with μSTDoF, which convert the "ordered" energy of the "macroscopic, deterministic" degrees of freedom into thermal agitation: these degrees of freedom are included in the system in order to keep track of the energy that abandons the Hamiltonian part for dissipation. Once the μSTDoF are included, the system dynamical variables are enlarged as $\mathbf{x} = (\mathbf{y}, \boldsymbol{\Sigma})$, with the vector $\boldsymbol{\Sigma}$ collecting the thermodynamic representation of the μSTDoF. When the system is complete the total energy is represented by the sum

$$H(\mathbf{y}, \boldsymbol{\Sigma}) = H_0(\mathbf{y}) + U(\mathbf{y}, \boldsymbol{\Sigma}): \tag{22}$$

the addendum $U(\mathbf{x})$ in general includes both a purely μSTDoF term, we would refer to as *internal energy*, and an interaction term depending on the whole configuration $\mathbf{x}$; for simplicity U may be supposed to depend only on the μSTDoF, and the equation (22) is rather re-written as

$$H(\mathbf{y}, \boldsymbol{\Sigma}) = H_0(\mathbf{y}) + U(\boldsymbol{\Sigma}) \tag{23}$$

(one may say that the Hamiltonian system of variables $\mathbf{y}$ and the μSTDoF of variables $\boldsymbol{\Sigma}$ are assumed to be *separable*).

Even if the formulation with Hamiltonian (22), or its simplified version (23), includes all the dynamical variables of the system, spanning the phase space M of complete configurations $\mathbf{x} = (\mathbf{y}, \boldsymbol{\Sigma})$, as long as it remains purely Hamiltonian no hope exists of seeing the system converge to an asymptotic equilibrium, as required instead for any *isolated system with dissipation*. This is why one needs to move ahead including properly "dissipative forces" into the formulation of Hamiltonian $H(\mathbf{y}, \boldsymbol{\Sigma})$ resorting a *metric component*, so that a metriplectic scheme is obtained. As metric systems are moved by a Lyapunov quantity, such an attribute of the system, monotonic with time due to dissipation, must be used.

For isolated systems with dissipation, Classical Thermodynamics predicts that dissipation is accompanied by the *increase of entropy* [7], a quantity that measures how much underdetermined the

microscopic configuration is once the macroscopic one is assigned [8]: the system of variables $\mathbf{x}=(\mathbf{y},\boldsymbol{\Sigma})$ must have a proper entropy, hence, and it is expected to grow monotonically. Actually the entropy is only attributed to the μSTDoF, which is its only part to be treated statistically: the total entropy S of the complete system will simply be the thermodynamic entropy of its μSTDoF. One then has:

$$S=S(\boldsymbol{\Sigma}).$$

Now, in order to define the metric part of the MBA, this S is of course the most obvious candidate to play the Lyapunov quantity Q in § 2.2: *the entropy of the μSTDoF must enter F as the metric contribution to the metriplectic generator*.

The construction of the MBA describing complete systems is then performed by considering as generating function a combination of H and S named *free energy*

$$F(\mathbf{y},\boldsymbol{\Sigma})=H(\mathbf{y},\boldsymbol{\Sigma})+\alpha S(\boldsymbol{\Sigma}), \tag{24}$$

where α is a parameter to be adjusted suitably: since everything's chosen in order for $H(\mathbf{y},\boldsymbol{\Sigma})$ to be constant, and for $S(\boldsymbol{\Sigma})$ to grow with time, the behavior of F with time depends on the sign of α:

$$\dot{F}=\alpha\dot{S} \quad \Rightarrow \quad \operatorname{sign}(\dot{F})=\operatorname{sign}(\alpha). \tag{25}$$

The assumption (24) turns (21) into:

$$\dot{f}=J^{ij}\partial_i f\partial_j H+\alpha J^{ij}\partial_i f\partial_j S+G^{ij}\partial_i f\partial_j H+\alpha G^{ij}\partial_i f\partial_j S. \tag{26}$$

The scheme is completed choosing J and G in (19) according to Physics. In general, the interaction between the original Hamiltonian system and the μSTDoF is tuned by some constant η, so that when $\eta\to 0$ the subsystems decouple and dissipation disappears. In this limit, clearly, the bracket $\langle\langle .,. \rangle\rangle$ must reduce to the original Poisson bracket $\{.,.\}$ moving $\mathbf{y}$ via $\{.,H_0\}$, in which only derivatives $\partial_{\mathbf{y}}$ with respect to $\mathbf{y}$ appear: the internal energy $U(\boldsymbol{\Sigma})$ is not affected by non-dissipative dynamics, so that one has $\{.,H_0\}=\{.,H\}$. All in all, the limit $\lim_{\eta\to 0}\langle\langle .,. \rangle\rangle=\{.,.\}$ must hold, so that, on the one hand the metric tensor G in (21) has to vanish for $\eta\to 0$

$$\lim_{\eta\to 0} G^{ij}=0,$$

and on the other hand the tensor J is simply the one forming the Poisson bracket of the Hamiltonian system with dynamical variables $\mathbf{y}$ and Hamiltonian H_0 we started from. In particular, J components pertaining to the sub-manifold described by $\boldsymbol{\Sigma}$ are zero.

When the metriplectic dynamics is enforced as $\dot{f}=J^{ij}\partial_i f\partial_j F+G^{ij}\partial_i f\partial_j F$, two facts must hold:

$$\dot{H}(\mathbf{y},\boldsymbol{\Sigma})=0, \quad \dot{S}(\boldsymbol{\Sigma})\geq 0.$$

The requirement of H to be constant with time is inserted into (26), giving rise to

$$0=\alpha J^{ij}\partial_i H\partial_j S+G^{ij}\partial_i H\partial_j H+\alpha G^{ij}\partial_i H\partial_j S,$$

where the anti-symmetry of J has been taken into account. On the other hand, the first addendum $\alpha J^{ij}\partial_i H\partial_j S$ is equal to $-\alpha J^{ab}\partial_a S\partial_b H=-\alpha\{S,H\}$, $\{S,H\}$ being the variability with time of S under the mere Hamiltonian part of the motion: now, since no change in the μSTDoF entropy is expected due

to the "conservative forces" represented by $\{.,H\}$, one expects to have $\alpha J^{ij}\partial_i H\partial_j S=0$. This is a precise request on the relationship between S and the symplectic part of $\langle\langle .,. \rangle\rangle$, and we'll discuss it soon; for the moment being, let's simply consider

$$\dot{H}=0 \quad \Rightarrow \quad 0=G^{ij}\partial_i H\left(\partial_j H+\alpha\partial_j S\right).$$

In order for the factor $G^{ij}\partial_i H\left(\partial_j H+\alpha\partial_j S\right)$ to be zero, the simplest thing to assume is

$$G^{ij}\partial_i H=0 \quad \Rightarrow \quad (H,f)=0 \quad \forall \quad f\in C^{\infty}(M,\mathbf{R}), \tag{27}$$

i.e. to assume that the metric tensor G has the gradient of H among its null vectors, so that the Hamiltonian has null metric bracket with any other element of $C^{\infty}(M,\mathbf{R})$. This (27) is what remains of the request that the total energy of the complete system is conserved.

About the monotonic increase of S, instead, while (27) holds, one may apply (21) and (24) and obtain:

$$\dot{S}=J^{ij}\partial_i S\partial_j H+\alpha G^{ij}\partial_i S\partial_j S.$$

Now, if a formal request

$$\{f,S\}=0 \quad \forall \quad f\in C^{\infty}(M,\mathbf{R}),$$

meaning that *S is a Casimir of the Poisson bracket at hand*, one simply gains:

$$\dot{S}=\alpha G^{ij}\partial_i S\partial_j S.$$

Choosing the sign of $\alpha\det G$ as positive, the condition $\dot{S}\geq 0$ is immediately satisfied. Considering $(.,H)=0$ and $\{.,S\}=0$ identically, the evolution of the system, and of any observable along the system trajectory, reads:

$$\dot{x}^i=\{x^i,H\}+\alpha(x^i,S), \quad \dot{f}=\{f,H\}+\alpha(f,S).$$

Thus, we end up with a dynamical theory referred to as *complete metriplectic system* (CMS).

3.1.2. Frictionless dissipation

The intervention of μSTDoF draining energy from deterministic variables **y** of a Hamiltonian system through friction may be not necessary in order for a CMS to admit asymptotically stable equilibria: there exist CMS the dynamics of which still undergoes the rules (18), (19) and (20) as:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt}=\{\mathbf{x},H(\mathbf{x})\}+\alpha(\mathbf{x},Q(\mathbf{x})), \\ \dot{Q}\geq 0, \\ \{Q,f\}=0, \quad (H,f)=0 \quad \forall \quad f, \end{cases}$$

in which the entropy-like observable Q and the Hamiltonian H depend on the same variables **x**. These CMS may be indicated as "frictionless" because μSTDoF cannot be singled out but, as the example in § 3.5, so the terms causing $\frac{d\mathbf{x}}{dt}=...$ to be a time-irreversible equation do not represent a "cascade" of energy from large time- or space-scales to smaller ones, but simply the intervention of the symmetric component $(.,.)$ of a Leibniz algebra generating the motion.

The "big difference" between CMS with friction and frictionless ones is that in the first case a granular nature of the material system must be supposed, so that dissipation transfers energy from "macroscopic" to "microscopic" scales: hence, a CMS with friction *must be non-elementary* in a sense, because the deterministic variable **y**, e.g. in (22), describe the macroscopic world averaging away fluctuations of microscopic constituents of matter, that will be encoded in the thermodynamic coordinates **Σ** of the μSTDoF. Only the μSTDoF may be "elementary" degrees of freedom, but they are treated statistically…

Frictionless CMS instead *implement dissipation at a fundamental level*, via the $\alpha(.,Q)$ component, which does not require any "smaller" constituent to exist. Simply, frictionless CMS seem to teach that irreversibility do not require the "coarse graining of micro-things", but rather the appearance of the semi-metric component of $\langle\langle .,. \rangle\rangle$, yielding a Lyapunov Q the growth of which just agrees with the verse of time flow.

The contribution of frictionless CMS at a fundamental level is two-folded: on the one hand, they generalize the idea of entropy $S(\mathbf{\Sigma})$ to Lyapunov dynamics generators $Q(\mathbf{x})$; on the other hand, their quantization may give hints to implement dissipation-irreversibility in terms of fundamental "microscopic" laws of Physics.

3.2. Stationary Points

The role of free energy F is, as it was happening in any Leibniz system, to provide with the stationary points of the dynamics at hand in terms of its extrema. Indeed, since $\dot{f} = \left(J^{ij}\partial_i f + G^{ij}\partial_i f\right)\partial_j F$ it is clear that nothing in the system will change if it is put in some configuration $\mathbf{x}_0$ so that $\partial_j F(\mathbf{x}_0) = 0$.

In the case of CMS with friction, due to the definition (24) and to the nature of the phase space coordinates $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$, the condition $\partial_j F(\mathbf{x}_0) = 0$ corresponds to the collection of conditions:

$$\begin{cases} \dfrac{\partial H}{\partial \mathbf{y}} = 0, \\ \dfrac{\partial H}{\partial \mathbf{\Sigma}} + \alpha \dfrac{\partial S}{\partial \mathbf{\Sigma}} = 0. \end{cases}$$

More precisely, considering the decomposition (23) of the Hamiltonian, one rather has to write:

$$\frac{\partial F}{\partial \mathbf{x}} = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial H_0}{\partial \mathbf{y}} = 0, \\ \dfrac{\partial U}{\partial \mathbf{\Sigma}} + \alpha \dfrac{\partial S}{\partial \mathbf{\Sigma}} = 0. \end{cases} \tag{28}$$

The relationship $\frac{\partial H_0}{\partial \mathbf{y}} = 0$ prescribes the *mechanical equilibrium* (in radiation systems this could be as well a "radiative" equilibrium, in that case **y** are field variables), while the relationship $\frac{\partial U}{\partial \mathbf{\Sigma}} + \alpha \frac{\partial S}{\partial \mathbf{\Sigma}} = 0$ is the *thermodynamic equilibrium* request. The latter will also help us fixing α in a physically sensible way, typically as minus the temperature of the μSTDoF.

About the nature of maxima or minima of the stationary points (28), one should calculate explicitly $\dot{F} = \langle\langle F, F \rangle\rangle$ considering (24), that would clearly re-lead to (25): in turn, it is sensible to establish α considering the physical sense of (28), and *then* assessing whether $\dot{F}$ has a positive or negative sign.

As far as the frictionless CMS are concerned, their dynamical generator F is still subdivided into a Hamiltonian "plus some Lyapunov"

$$F(\mathbf{x}) = H(\mathbf{x}) + \alpha Q(\mathbf{x});$$

however, as long as the decomposition $\mathbf{x} = (\mathbf{y}, \mathbf{\Sigma})$ is not available, the steady state prescription

$$\frac{\partial F}{\partial \mathbf{x}} = 0 \tag{29}$$

will have a different interpretation; in some cases, as the one discussed in § 3.5, the determination of α out of (29) still leads to equating it to some (sign reverted) temperature.

3.3. Two Elementary Examples with Friction

Two very simple examples can be given, taken from the Newtonian mechanics, and very useful to start getting acquainted with CMS.

3.3.1. The point particle in the viscous medium

In the first example the complete system is formed by a point particle of mass m moving in the 3-dimensional space through an infinite viscous medium [3], so that its equations of motion read

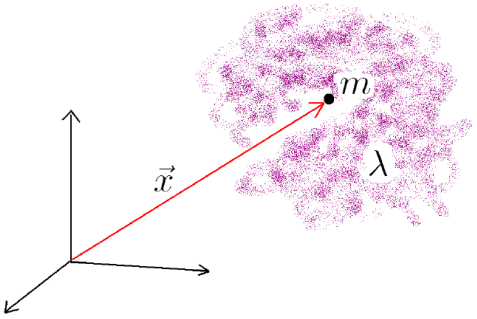
$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}, \quad \frac{d\vec{p}}{dt} = -\frac{\partial V}{\partial \vec{x}} - \frac{\lambda}{m}\vec{p}. \tag{30}$$

The foregoing equations would reduce to a Hamiltonian system $\frac{d}{dt}\vec{x} = m^{-1}\vec{p}$ and $\frac{d}{dt}\vec{p} = -\partial_{\vec{x}} V$ in the non-dissipative limit $\lambda \to 0$, the dynamical variables of which would be $\mathbf{y} = (\vec{x}, \vec{p})$, and with Hamiltonian $H_0(\mathbf{y}) = \frac{1}{2m}p^2 + V(\vec{x})$. In order to recognize a metriplectic complete system one has to include the viscous medium variables, introducing a very simple collection of (thermo)dynamical coordinates, namely the medium entropy only $\mathbf{\Sigma} = (S)$: hence the whole Hamiltonian reads

$$H = H_0(\mathbf{y}) + U(S),$$

with internal energy of the viscous medium $U(S)$. Of course, the complete dynamical variable is $\mathbf{x} = (\vec{x}, \vec{p}, S)$.

Figure 1. The point particle of mass m and position $\vec{x}$ moving through the medium of viscous constant λ, described as a metriplectic system in § 3.3.1.



The cartoon of the system is that of Figure 1.

In order to construct the MBA for equations (30) it is necessary to know the ODE for the entropy, which is obtained by the classical relationship $đQ = TdS$, being T the temperature of the medium and $đQ$ the amount of thermal energy transferred from the mechanical degrees of freedom **y** to the viscous medium. Actually, since the energy must be conserved, $đQ$ is simply minus the power of the dissipative force $-m^{-1}\lambda\vec{p}$ exerted by the medium on the point particle, times the infinitesimal time interval dt: this power being $w_\lambda = -m^{-1}\lambda\vec{p}\cdot m^{-1}\vec{p}$ and $đQ = -w_\lambda dt$, one has $đQ = m^{-2}\lambda p^2 dt$, and considering $dS = \dot{S}dt$ one ends up with:

$$\dot{S} = \frac{\lambda p^2}{m^2 T} \tag{31}$$

(obviously, the temperature of the medium is supposed to remain constant, while friction transfers such a small amount of energy from the pointlike particle). Equation (31) completes the ODEs of the complete system we are looking for, together with equations (30).
According to equation (24), the free energy of the system reads

$$F(\vec{x},\vec{p},S) = \frac{p^2}{2m} + V(\vec{x}) + U(S) + \alpha S : \tag{32}$$

the parameter α may be determined by imposing that the extrema of F represent steady states. The steady state $\mathbf{x}_0$ is found as the solution of

$$\frac{\partial F}{\partial \vec{x}} = 0, \quad \frac{\partial F}{\partial \vec{p}} = 0, \quad \frac{\partial F}{\partial S} = 0. \tag{33}$$

If (32) is put into (33) one finds

$$\frac{\partial V}{\partial \vec{x}}(\vec{x}_0) = 0, \quad \vec{p}_0 = 0, \quad \frac{\partial U}{\partial S}(S_0) + \alpha = 0 :$$

the first and second equations mean that the point particle will stop at an extremum of the mechanical potential, with null momentum (velocity); the third equation, better re-written as $\alpha = -\frac{\partial U}{\partial S}(S_0)$, instead determines α as minus $\frac{\partial U}{\partial S}(S_0)$, which is clearly *the equilibrium temperature of the viscous medium T*, after the relationship $\frac{\partial U}{\partial S} = T$ due to Classical Thermodynamics [7]. Equilibrium equations

$$\frac{\partial V}{\partial \vec{x}}(\vec{x}_0) = 0, \quad \vec{p}_0 = 0, \quad \alpha = -T$$

allow to re-write the free energy (32) in the following way:

$$F(\vec{x},\vec{p},S) = \frac{p^2}{2m} + V(\vec{x}) + U(S) - TS. \tag{34}$$

By the way, due to the relationship $\alpha = -T$, the tensor G has to be negative semi-definite.

In order to complete the MBA one has to define suitable J and G. The tensors will be written with respect to the components $\vec{x}$, $\vec{p}$ and S, so they will look like:

$$J=\begin{pmatrix}\mathbf{0}_{3,3} & \mathbf{1}_{3,3} & \mathbf{0}_{3,1}\\ -\mathbf{1}_{3,3} & \mathbf{0}_{3,3} & \mathbf{0}_{3,1}\\ \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & 0\end{pmatrix},\quad G=\frac{1}{\alpha}\begin{pmatrix}\dfrac{\left|\partial_{\vec{x}}V\right|^2\mathbf{1}_{3,3}-\partial_{\vec{x}}V\otimes\partial_{\vec{x}}V}{\left|\partial_{\vec{x}}V\right|^2} & \mathbf{0}_{3,3} & \mathbf{0}_{3,1}\\ \mathbf{0}_{3,3} & \lambda T\mathbf{1}_{3,3} & -\lambda m^{-1}\vec{p}\\ \mathbf{0}_{1,3} & -\lambda m^{-1}\vec{p}^{\,T} & \dfrac{\lambda p^2}{m^2T}\end{pmatrix}. \tag{35}$$

In (35), note that G has the same definition as the sign of α, since the matrix multiplying α^{-1} has either null eigenvector. With the matrices defined in (35) the relationships

$$\{f,S\}=0,\quad (f,H)=0$$

for any f are satisfied, moreover one obtains

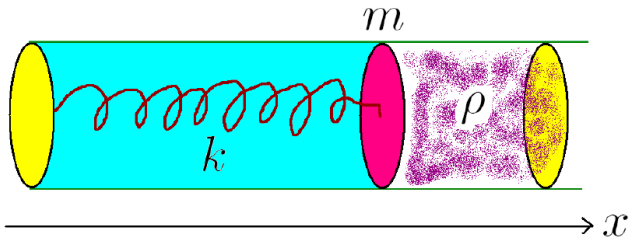
$$\{\vec{x},H\}+\alpha(\vec{x},S)=\frac{\vec{p}}{m},\quad \{\vec{p},H\}+\alpha(\vec{p},S)=-\frac{\partial V}{\partial\vec{x}}-\frac{\lambda}{m}\vec{p},\quad \alpha(S,S)=\frac{\lambda p^2}{m^2T},$$

meaning that we are in the presence of the correct tensors (35) to define the MBA reproducing the ODEs (30) and (31). All in all, with the tensors in (35) and the generating function F in (34), we can state: $\dot{f}=\langle\langle f,F\rangle\rangle$ for any observable f of the system.

3.3.2. The piston and the spring

The second example of metriplectic system taken from everyday Physics is a piston of mass m and surface A, moving along a horizontal guide and pushed by a spring of constant k. This piston makes work against a viscous gas of mass M, the pressure of which is indicated as P. This system is depicted as in Figure 2.

Figure 2. The piston and the spring, described in § 3.3.2 as a metriplectic complete system. The density of the gas is indicated as ρ.



If no viscosity were present, the system would be conservative, i.e. Hamiltonian: the necessary, independent dynamical variables would just be x and p of the piston; when energy is irreversibly transferred between the mechanical degrees of freedom and the μSTDoF of the gas, due to viscosity, some thermodynamic coordinate of the medium must be included: the entropy S of it is the simplest candidate, so that in this case the dynamical variables of the complete system are collected in the vector $\mathbf{x}=(x,p,S)$. The ODEs of $\mathbf{x}$ are written as [3]:

$$\dot{x}=\frac{p}{m},\quad \dot{p}=-k(x-\ell)-PA-\lambda\frac{p}{m},\quad \dot{S}=\frac{\lambda p^2}{m^2T} \tag{36}$$

(the parameter ℓ is the equilibrium length of the spring, while T is the temperature of the gas, and the same reasoning about $đQ = TdS$ as in § 3.3.1 has been applied). As far as the thermodynamics of the gas is concerned, this case is more interesting than the one discussed above, where the point particle could vary nothing of the infinite medium. In general, indeed, one may expect that the thermodynamic coordinates of the gas should be S and, e.g., the mass density ρ, so that its internal energy reads $U(\rho, S)$, and the pressure P in (36) is defined as

$$PV = \rho \frac{\partial U}{\partial \rho} \quad \Rightarrow \quad P = \frac{\rho^2}{M} \frac{\partial U}{\partial \rho}. \tag{37}$$

Here the only fixed things of the gas are the area A of the piston and the mass M of the medium: in general, its volume reads $V(x) = A(L_0 - x)$, so that the density will depend on the position of the piston as

$$\rho(x) = \frac{M}{A(L_0 - x)} \quad \Rightarrow \quad \frac{\partial x}{\partial \rho} = \frac{M}{A\rho^2}, \quad \frac{\partial \rho}{\partial x} = \frac{A\rho^2}{M}. \tag{38}$$

This x-dependence in the density does imply that the coordinates of the complete system are (x, p, S) instead of the redundant set (x, p, ρ, S); it also yields the dependence $U(\rho(x), S)$, so that the subsystems "piston-attached-to-the-spring" and "gas" *are not separable*.

The whole Hamiltonian of the system includes the kinetic energy of the piston, the elastic energy of the spring and the internal energy of the viscous gas $U(\rho(x), S)$. Once it's written, it is very easy to write also the free energy F:

$$\begin{aligned} H(x, p, S) &= \frac{p^2}{2m} + \frac{k}{2}(x - \ell)^2 + U(\rho(x), S), \\ F(x, p, S) &= \frac{p^2}{2m} + \frac{k}{2}(x - \ell)^2 + U(\rho(x), S) + \alpha S. \end{aligned} \tag{39}$$

In order for the generator $F(x, p, S)$ to produce the correct ODEs (36), one just has to choose the two tensors J and G as follows

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda T & -\lambda m^{-1} p \\ 0 & -\lambda m^{-1} p & \dfrac{\lambda p^2}{m^2 T} \end{pmatrix},$$

being written with respect to the dynamical variables (x, p, S). Last but not least, one may find out the equilibria of the system by checking the extrema of the function F as:

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial p} = 0, \quad \frac{\partial F}{\partial S} = 0.$$

With the specific form (39), and considering the relationship among U, P, x and ρ in (37) and (38), one obtains

$$x_0 = \ell - \frac{PA}{k}, \quad p_0 = 0, \quad \alpha = -T:$$

respectively, these are the balance between the gas and the spring forces, the zero velocity of the piston and the correspondence of the parameter α to minus the temperature of the gas.

3.4. Classical Fluids

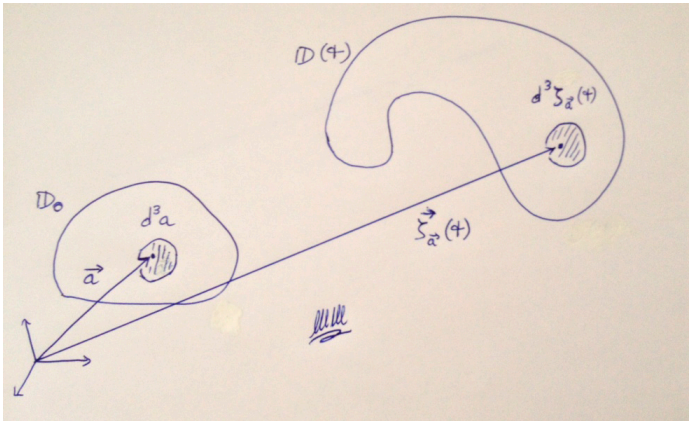
In fluid theories, in the absence of dissipation the energy transfer would take place among degrees of freedom making sense at a macroscopic scale, and ignoring the granular nature of matter; however, as "friction" is turned on, by considering finite viscosity, the degrees of freedom of the microscopic particles forming the fluid play the role of the μSTDoF discussed before.

About this, a remark is necessary, to stress the difference between these systems and those discussed in § 3.3. In the examples in § 3.3 the complete system was subdivided into *two subsystems "materially" separated*: in § 3.3.1 there was a point particle treated deterministically and a viscous fluid with μSTDoF, while in § 3.3.2 the deterministic degrees of freedom were those of the piston, while μSTDoF were attributed to the viscous gas against which the piston was working. Dissipative continua do not show a "material separation" between the deterministic, Hamiltonian part of the complete system and the μSTDoF draining energy irreversible and, hence, giving rise to dissipation. Consider a "macroscopic infinitesimal" *parcel*, i.e. a portion of fluid containing a thermodynamic number of particles and still being so small that the continuum field variables are constant within it: the motion of the particles within the parcel represent the μSTDoF, while the motion of the parcel's center-of-mass (CoM) throughout the space represent the degrees of freedom **y** in § 3.1, with a deterministic dynamics. In a fluid context, the energy pertaining to the center-of-mass of a given parcel δC is irreversibly converted, by dissipation, into the energy of the μSTDoF relative to another parcel $\delta C'$ (δC and $\delta C'$ are *different parcels*, otherwise a mechanical system would be able to alter its own CoM motion, against Newton's Principles: this is why dissipative terms appear in the equations of motion of fluid dynamics with space-derivative terms, taking into account of the different CoM velocities of nearby parcels [9]. With the dissipation due to currents the thing is slightly subtler, because the energy irreversibly converted there comes from the magnetic degrees of freedom [4, 10, 11]).

Dissipative fluids may be understood as complete metriplectic systems in which the **y** and the **Σ** just describe *different degrees of freedom of the same material system*: as explained before, *these variables describe the system at different space- and time-scales*.

In the Lagrangian representation of dissipative fluids the dynamical variables are the CoM position $\vec{\zeta}(\vec{a})$ and momentum $\vec{\pi}(\vec{a})$ of each fluid parcel, plus the variables relative-to-the-CoM of the same parcels, thermodynamically described through the entropy density $s(\vec{a})$. The 3D continuous index $\vec{a} \in \mathbf{D}_0$ materially labels the fluid parcels [12], it refers to the initial position of the parcel that labels as $\vec{a} = \vec{\zeta}(\vec{a},0)$, if $\vec{\zeta}(\vec{a},t)$ is the position of the $\vec{a}$-th parcel at time *t*. The domain $\mathbf{D}_0$ is the initial volume occupied by the continuous matter.

Figure 3. A cartoon by the Author showing the idea of Lagrangian (material) coordinates for the continuum system: from the 3D volume $\mathbf{D}_0$, initially occupied by the fluid matter, the system evolves into some volume $\mathbf{D}(t)$ it occupies at time t. While the $\mathbf{D}_0$ is spanned by the 3D label $\vec{a}$, at time t the volume $\mathbf{D}(t)$ is described by the positions $\vec{\zeta}(\vec{a},t)$ that are the transformed positions of the fluid parcels, namely being $\vec{a}=\vec{\zeta}(\vec{a},0)$. The initial parcel volume d^3a gets transformed into $d^3\zeta(\vec{a},t)$ at successive times, with the relationship $d^3\zeta(\vec{a},t)=\mathrm{J}(\vec{a},t)d^3a$, see text for details. Note that the $\vec{a}$-dependence is indicated via a subscript in the Figure, so one must understand $d^3\zeta_{\vec{a}}(t)\equiv d^3\zeta(\vec{a},t)$ and so on.



The subdivision between "Hamiltonian" variables $\mathbf{y}$ and μSTDoF variables $\mathbf{\Sigma}$ is simple, $\mathbf{y}=\left(\vec{\zeta}(\vec{a}),\vec{\pi}(\vec{a})\right)$, $\mathbf{\Sigma}=(s(\vec{a}))$, while the mass $\rho(\vec{a})$ of the density $\vec{a}$-th parcel may be expressed in terms of the initial mass density $\rho_0(\vec{a})$, that characterizes the mass geometry of the continuum and is assigned once and forever, and the Jacobian determinant $\mathrm{J}(\vec{a})=\det\left\|\frac{\partial\vec{\zeta}}{\partial\vec{a}}\right\|$, that is a functional of the dependence of $\vec{\zeta}(\vec{a})$ on its parcel-index $\vec{a}$:

$$\rho(\vec{a})=\frac{\rho_0(\vec{a})}{\mathrm{J}(\vec{a})};$$

$\rho(\vec{a})$ would be redundant, as a dynamical variable, to the complete configuration $\mathbf{x}=\left(\vec{\zeta}(\vec{a}),\vec{\pi}(\vec{a}),s(\vec{a})\right)$.

Excluding dissipative forces, the infinitesimal parcel described by the configuration $\mathbf{x}$ can be attributed a Hamiltonian dynamics by considering its energy determined by the sum of a kinetic part, a potential part giving rise to the pressure forces exerted by the surrounding parcels, and a part due to external "conservative" forces. The total Hamiltonian of the fluid reads:

$$H\left[\vec{\zeta},\vec{\pi},s\right]=\int_{\mathbf{D}_0}d^3a\left[\frac{\pi^2(\vec{a})}{2\rho_0(\vec{a})}+\rho_0(\vec{a})U\left(\frac{\rho_0(\vec{a})}{\mathrm{J}(\vec{a})},s(\vec{a})\right)+\rho_0(\vec{a})V\left(\vec{\zeta}(\vec{a})\right)\right]. \tag{40}$$

The dissipation free motion of the fluid is generated by the foregoing Hamiltonian and the Poisson bracket:

$$\{f,g\}=\int_{\mathbf{D}_0}d^3a\left[\frac{\delta f}{\delta\vec{\zeta}(\vec{a})}\cdot\frac{\delta g}{\delta\vec{\pi}(\vec{a})}-\frac{\delta g}{\delta\vec{\zeta}(\vec{a})}\cdot\frac{\delta f}{\delta\vec{\pi}(\vec{a})}\right]. \tag{41}$$

The Hamiltonian limit of the fluid dynamics is then:

$$\dot{f}\left[\vec{\zeta},\vec{\pi},s\right]=\left\{f\left[\vec{\zeta},\vec{\pi},s\right],H\left[\vec{\zeta},\vec{\pi},s\right]\right\}. \tag{42}$$

The symplectic bracket (41) is a *canonical one*, not different from those of ordinary point particles of Newtonian Physics [5]. In particular, *it doesn't contain any derivative with respect to the entropy density* $s(\vec{a})$, so it does not involve any statistical proxy of the μSTDoF: this is a benefit brought by the Lagrangian representation, while in the Eulerian one derivatives with respect to ρ and s appear [11]. Of course, microscopic degrees of freedom of the parcel are present in $H\left[\vec{\zeta},\vec{\pi},s\right]$, in particular through the internal potential energy $U\left(\frac{\rho_0(\vec{a})}{\mathrm{J}(\vec{a})},s(\vec{a})\right)$: still, due to the form of (41), the entropy density $s(\vec{a})$ is perfectly conserved

$$\dot{s}(\vec{a})=\left\{s(\vec{a}),H\left[\vec{\zeta},\vec{\pi},s\right]\right\}=0:$$

as the ideal, dissipation-free motion of the fluid takes place, the amount $s(\vec{a})$, encoding the complexity of the particle motions internal to the macroscopic parcel, remains frozen to its original value. One could well consider $s(\vec{a},t)=s(\vec{a},0)\equiv s_0(\vec{a})$ a parameter rigidly assigned (as the density $\rho_0(\vec{a})$), so that the Hamiltonian (40) would become a quantity as the H_0 in (23). In our case, the "free Hamiltonian" $H_0(\mathbf{y})$ would rather read

$$H_0\left[\vec{\zeta},\vec{\pi}\right]=\int_{\mathbf{D}_0}d^3a\left[\frac{\pi^2(\vec{a})}{2\rho_0(\vec{a})}+\rho_0(\vec{a})U\left(\frac{\rho_0(\vec{a})}{\mathrm{J}(\vec{a})},s_0(\vec{a})\right)+\rho_0(\vec{a})V\left(\vec{\zeta}(\vec{a})\right)\right].$$

In the foregoing formula H_0 just depends on the CoM variables because the other ones are frozen and "do not exist" as dynamical variables. Considering the prescription (42), the fluid dynamics in Lagrangian variables is written as:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\ \dot{\pi}_\alpha=-\rho_0\dfrac{\partial V}{\partial\zeta^\alpha}+A^i_\alpha\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathrm{J}}\right),\quad A^i_\alpha=\dfrac{\varepsilon_{\alpha\kappa\lambda}\varepsilon^{imn}}{2}\dfrac{\partial\zeta^\kappa}{\partial a^m}\dfrac{\partial\zeta^\lambda}{\partial a^n},\\ \dot{s}=0\end{cases} \tag{43}$$

(Greek indices refer to the vectors $\vec{\zeta}$ and $\vec{\pi}$, Latin ones to the 3D parcel-index $\vec{a}$). In the ideal case without dissipation the density, s has zero Poisson bracket "with anything"

$$\left\{f,s(\vec{a})\right\}=0\quad\forall\quad f, \tag{44}$$

so does the total entropy of the fluid defined in (46), see below.

Including the interaction with the μSTDoF means simply unfreeze the thermodynamic quantity $s(\vec{a})$ thanks to dissipative "forces" that may change its value. The dissipative forces we are talking about would turn the foregoing equations (43) into the following ones:

$$\begin{cases}\dot{\zeta}_\alpha=\pi_\alpha,\\ \dot{\pi}_\alpha=-\rho_0\dfrac{\partial V}{\partial\zeta^\alpha}+A^i_\alpha\dfrac{\partial}{\partial a^i}\left(\rho_0\dfrac{\partial U}{\partial\mathrm{J}}\right)+\Lambda_{\alpha\beta\gamma\delta}\mathrm{J}\nabla^\beta\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right),\\ \dot{s}=\dfrac{\mathrm{J}}{\rho_0 T}\Lambda_{\alpha\beta\gamma\delta}\nabla^\alpha\left(\dfrac{\pi^\beta}{\rho_0}\right)\nabla^\gamma\left(\dfrac{\pi^\delta}{\rho_0}\right)+\dfrac{\mathrm{J}\kappa}{\rho_0 T}\nabla^\alpha\nabla_\alpha T.\end{cases} \tag{45}$$

The convention $\vec{\nabla} = \frac{\partial}{\partial \vec{\zeta}}$ is intended. In (45) the coefficient κ is the thermal conductivity, while T is the temperature of the μSTDoF within the parcel, defined as always: $T(\vec{a}) = \frac{\partial U}{\partial s(\vec{a})}$. As far as the symbol $\Lambda_{\alpha\beta\gamma\delta}$ is concerned, we simply state:

$$\Lambda_{\alpha\beta\gamma\delta} = \eta\left(\delta_{\delta\alpha}\delta_{\gamma\beta} + \delta_{\delta\beta}\delta_{\gamma\alpha} - \frac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\delta} \right) + \zeta\delta_{\alpha\beta}\delta_{\gamma\delta}.$$

In (45) the entropy variation is due to the non-ideal stress tensor, hence viscosity

$$\frac{\mathrm{J}}{\rho_0 T}\Lambda_{\alpha\beta\gamma\delta}\nabla^{\alpha}\left(\frac{\pi^{\beta}}{\rho_0}\right)\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right),$$

and to thermal conduction

$$\frac{\mathrm{J}\kappa}{\rho_0 T}\nabla^{\alpha}\nabla_{\alpha}T,$$

i.e. to the two irreversible processes taking place. These are the processes draining energy out of the parcels' CoM degrees of freedom $\mathbf{y} = \left(\vec{\zeta}(\vec{a}), \vec{\pi}(\vec{a})\right)$, smoothing the velocity difference between nearby parcels (i.e. killing the gradients $\partial_{\vec{a}}\vec{\pi}$) and homogenizing temperature T via thermal diffusion.

It is possible to construct a functional derivative semi-metric bracket that defines a MBA together with the Poisson bracket (41), that can reproduce equations (45). As usual, the entropy of the μSTDoF responsible for dissipation must be introduced

$$S[s] = \int_{\mathbf{D}_0} d^3a\,\rho_0(\vec{a})s(\vec{a}), \tag{46}$$

that is a Casimir of (42), as its integrand is, see equation (44), and composes the free energy together with the Hamiltonian:

$$F\left[\vec{\zeta}, \vec{\pi}, s\right] = H\left[\vec{\zeta}, \vec{\pi}, s\right] + \alpha S[s]. \tag{47}$$

Then, the symmetric bracket completing the MBA reads [9]:

$$(f,g) = \frac{1}{\alpha}\int_{\mathbf{D}_0}\mathrm{J}d^3a\left\{T\Lambda_{\alpha\beta\gamma\delta}\left[\nabla^{\alpha}\left(\frac{\delta f}{\delta\pi_{\beta}}\right) - \frac{1}{\rho_0 T}\nabla^{\alpha}\left(\frac{\pi^{\beta}}{\rho_0}\right)\frac{\delta f}{\delta s}\right]\left[\nabla^{\gamma}\left(\frac{\delta g}{\delta\pi_{\delta}}\right) - \frac{1}{\rho_0 T}\nabla^{\gamma}\left(\frac{\pi^{\delta}}{\rho_0}\right)\frac{\delta g}{\delta\sigma}\right] + \right.$$
$$\left. + \kappa T^2\nabla^{\alpha}\left(\frac{1}{\rho_0 T}\frac{\delta f}{\delta s}\right)\nabla_{\alpha}\left(\frac{1}{\rho_0 T}\frac{\delta g}{\delta s}\right)\right\}.$$

The Hamiltonian (40) has null semi-metric bracket with anything, while the increase of entropy is given by putting together the definition (46) and the third equation in (45).

With this semi-metric bracket (that has the same definition of α, in terms of the sign) equations (45) are reproduced by assigning the usual metriplectic dynamics $\dot{f} = \{f, H\} + \alpha(f, S)$ for any physical functional $f\left[\vec{\zeta}, \vec{\pi}, s\right]$.

3.5. Kinetic Theory

The CMS described until now are essentially formed by a Hamiltonian system of variables **y** plus a "thermal bath" of statistical variables **Σ**, indicated as μSTDoF. The interaction between the two sub-

manifolds of the phase space M is described by the metric part of the Leibniz bracket algebra $\langle\langle .,. \rangle\rangle$ generating dynamics. In this session, then, we will make the example of a dynamical system that instead has only one dynamical variable and, still, may admit either a Hamiltonian description or a metriplectic one depending on whether dissipative, time-asymmetric interactions are included or not: together with the example of § 3.6 this case permits us to introduce a subtler distinction between the Hamiltonian and the non-Hamiltonian part of a MBA, namely the time-reversible and the time-irreversible one, crucial for the construction illustrated in § 4.

The example we are going to treat here is that of *kinetic theories*.

Kinetic theories represent a system of many identical particles through the so called *Boltzmann distribution* in the μ-space [13], i.e. a single particle space $\mathbf{R}^6$ in which $f = f(\vec{x},\vec{v},t)$ is the probability density that a particle of the system picked at random is at the position $\vec{x}$ in the 3D space, with velocity $\vec{v}$ at time t. We are discussing the so called Vlasov-Poisson system: a gas of electrically charged particles is coupled to an electrostatic field represented by its scalar potential $\phi(\vec{x})$ acting on the charged particles; then, the equation of motion of the μ-space distribution $f(\vec{x},\vec{v},t)$ is the Vlasov equation coupled with the Poisson equation for the electrostatic field [6]:

$$\partial_t f + \frac{\partial f}{\partial \vec{x}} \cdot \vec{v} - \frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial \phi}{\partial \vec{x}}[f] = W_{\text{coll}}[f]. \tag{48}$$

In the equation (48) the term $-\partial_{\vec{x}}\phi[f]$ represents the electrostatic field that, at any time, depends on the distribution f itself through the charge distribution in the space, while $W_{\text{coll}}[f]$ is referred to as the collision term representing the time variation of $f(\vec{x},\vec{v},t)$ due to *two-particle collisions* at that given point $\vec{x}$ at that time. The term $\phi[f]$ is constructed as a functional of f, rendering (48) an integro-differential equation:

$$\phi(\vec{x},t) = \int d^3v' \int d^3x' V(|\vec{x}-\vec{x}'|) f(\vec{x}',\vec{v}',t).$$

The kernel $V(|\vec{x}-\vec{x}'|)$ is simply the electrostatic potential in a point $\vec{x}$ due to the presence of the point particle at the position $\vec{x}'$: this determines the forces through which particles sense each other.

Kinetic theory states that equations as (48) are non-time-reversible due to the term $W_{\text{coll}}[f]$, that in practice represents the dissipative term giving rise to the increase of *entropy* (the so called *Boltzmann's H-Theorem*): the very beautiful thing is that, in the *collisionless limit* $W_{\text{coll}}[f] \to 0$, what remains of (44), i.e.

$$\partial_t f + \frac{\partial f}{\partial \vec{x}} \cdot \vec{v} - \frac{\partial f}{\partial \vec{v}} \cdot \frac{\partial \phi}{\partial \vec{x}}[f] = 0, \tag{49}$$

is a *Hamiltonian dynamical system*, while the collisional Vlasov-Poisson system (48) is represented as a CMS.

In practice, the *dissipative term* in (48), i.e. *2-particle collision term*, does determine the non-Hamiltonian, *semi-metric contribution* to the dynamical system and, needless to say, is moved *via a symmetric bracket by an entropic functional*.

Consider, first of all, the functional

$$H[f] = \int d^3x \int d^3v f(\vec{x},\vec{v}) \frac{mv^2}{2} + \frac{1}{2} \int d^3x \int d^3v \phi(\vec{x},t) f(\vec{x},\vec{v}) \tag{50}$$

and the Leibniz functional bracket

$$\{A[f],B[f]\}=\int d^3x\int d^3v f(\vec{x},\vec{v})\left(\partial_{\vec{x}}\frac{\delta A}{\delta f(\vec{x},\vec{v})}\cdot\partial_{\vec{v}}\frac{\delta B}{\delta f(\vec{x},\vec{v})}-\partial_{\vec{x}}\frac{\delta B}{\delta f(\vec{x},\vec{v})}\cdot\partial_{\vec{v}}\frac{\delta A}{\delta f(\vec{x},\vec{v})}\right): \tag{51}$$

not only this $\{.,.\}$ in (51) is a Poisson bracket satisfying anti-symmetry and Jacobi identity, but it also generates the collisionless equation (49) once the Hamiltonian functional $H[f]$ defined in (50) is made use of:

$$\partial_t f(\vec{x},\vec{v})=\{f(\vec{x},\vec{v}),H[f]\}.$$

In order to turn on collisions, and then obtain the equation (44), one resorts *Boltzmann's entropy*

$$S[f]=-k\int d^3x\int d^3v f(\vec{x},\vec{v})\ln f(\vec{x},\vec{v}): \tag{52}$$

as soon as one defines a *symmetric, semi-metric functional bracket*

$$\begin{aligned}&(A[f],B[f])=\\&=-\frac{L}{2\alpha}\int d^3x\int d^3v\int d^3x'\int d^3v'\left(\frac{\partial}{\partial v^i}\frac{\delta A}{\delta f(\vec{x},\vec{v})}-\frac{\partial}{\partial v'^i}\frac{\delta A}{\delta f(\vec{x}',\vec{v}')}\right)*\\&\left(\frac{\partial}{\partial v^i}\frac{\delta B}{\delta f(\vec{x},\vec{v})}-\frac{\partial}{\partial v'^i}\frac{\delta B}{\delta f(\vec{x}',\vec{v}')}\right)f(\vec{x},\vec{v})f(\vec{x},\vec{v}')\left[\frac{\delta^{ij}}{|\vec{v}-\vec{v}'|}-\frac{(v^i-v'^i)(v^j-v'^j)}{|\vec{v}-\vec{v}'|^3}\right]\delta^3(\vec{x}-\vec{x}'),\end{aligned} \tag{53}$$

one may rather easily check:

$$W_{\text{coll}}[f;\vec{x},\vec{v})=\alpha(f(\vec{x},\vec{v}),S[f]),$$

provided the collisional term $W_{\text{coll}}[f;\vec{x},\vec{v})$ is assumed of the form [6]:

$$\begin{aligned}&W_{\text{coll}}[f;\vec{x},\vec{v})=\\&=L\frac{\partial}{\partial v^i}\left\{\int d^3v'\left[\frac{\delta^{ij}}{|\vec{v}-\vec{v}'|}-\frac{(v^i-v'^i)(v^j-v'^j)}{|\vec{v}-\vec{v}'|^3}\right]\left[f(\vec{x},\vec{v})\frac{\partial f(\vec{x},\vec{v}')}{\partial v'^j}-f(\vec{x},\vec{v}')\frac{\partial f(\vec{x},\vec{v})}{\partial v^j}\right]\right\}.\end{aligned}$$

Considering the full free energy functional and the MBA constructed through (51) and (53)

$$F[f]=H[f]+\alpha S[f],\quad \langle\langle A,B\rangle\rangle=\{A,B\}+(A,B), \tag{54}$$

the equation (44) is finally reproduced:

$$\partial_t f(\vec{x},\vec{v})=\langle\langle f(\vec{x},\vec{v}),F[f]\rangle\rangle \tag{55}$$

(consider the Hamiltonian to be a null mode of the semi-metric component $(H[f],.)=0$, and the entropy to be a Casimir of the Poisson component $\{S[f],.\}=0$).

Before going to the next example, something should be stressed about the model (55), so to have a hint for the next § 4 and for concluding remarks.

As cleverly shown in [6], the model (55) may be adapted to make the system relax to different equilibrium field configurations $f_0(\vec{x},\vec{v})$, and the "tailoring" must be done on the entropic functional S in (52): since any functional of the form $S[f]=\int d^3x\int s(f(\vec{x},\vec{v}))d^3v$ is a Casimir of the Poisson bracket (51), to each function $s\in C^\infty(\mathbf{R},\mathbf{R})$ there will correspond a MBA, relaxing to a suitable

$f_0(\vec{x},\vec{v})$ thanks to the action of S through $\alpha(.,S[f])_s$, being $(.,.)_s$ a semi-metric functional bracket "tailored" on the function s. For instance, the S in (52) leads to the Boltzmann equilibrium with absolute temperature $T=-\alpha$, while if the function $s(f)=-k[f\ln f+(1-f)\ln(1-f)]$ were used, with a suitably adapted bracket $(.,.)_s$ (see [6] for details), then $f_0(\vec{x},\vec{v})$ would be a Fermi-Dirac one, still with absolute temperature given by the constant α appearing in (54) as $T=-\alpha$.

The other crucial observation on (54) and (55) is that the CMS at hand has one field variable only, i.e. the distribution *f*, so *one does wonder where the "Hamiltonian" sub-system (i.e.* ***y****) and the μSTDoF (i.e.* ***Σ****) are!* The answer, contained in $\dot{A}[f]=\{A[f],H[f]\}+\alpha(A,S[f])$, is that indeed *there does not exist* any frictionless sub-system and any μSTDoF draining "ordered" energy from it, *there are not fundamental sub-systems*: rather, *there exist fundamental algebraic sub-structures*, i.e. components of the algebraic structure of dynamics, namely $\{.,.\}$ and $(.,.)$, apparently giving rise to so deeply different behaviors with respect to time flow [14], i.e. "eternal perfect conservative Hamiltonian" and "ageing time-irreversible entropic" evolutions. The example in § 3.6 will clarify this point further.

3.6. Morrison's rotator

What is referred to as Morrison's rotator here is a toy model very similar, to many extents, to the kinetic metriplectic theory described in § 3.5, because the system has only one dynamical variable, namely the angular momentum $\vec{L}\in\mathbf{R}^3$ of a rigid body of the Newtonian Physics.

The free rigid rotator has equations of motion, written in terms of the components of $\vec{L}$ in the principal inertial axes, given by:

$$\dot{L}^i=\frac{\varepsilon^i{}_{jk}L_jL_k}{I^{(i)}}, \tag{56}$$

where $I^{(a)}$ is the momentum of inertia with respect to the *a*-th principal axis, along which the component of $\vec{L}$ reads L^a. The symbol $\varepsilon^i{}_{jk}$ is the partially contravariant form of Ricci $SO(3)$-tensor. The ODEs (56) are straightforwardly reproduced by the Poisson bracket

$$\{f,g\}=-\varepsilon_i{}^{jk}L^i\frac{\partial f}{\partial L^j}\frac{\partial g}{\partial L^k}, \tag{57}$$

provided the mechanical energy of the rigid rotator $\frac{1}{2}\vec{L}\cdot\vec{\omega}$ is used as a Hamiltonian, being $\vec{\omega}$ the angular velocity of the system, with $L^a=I^{(a)}\omega^a$. If the diagonal tensor of inertia $\sigma^{ab}=I^{(a)}\delta^{ab}$ is used, the Hamiltonian giving (56) thanks to (57) reads:

$$H(\vec{L})=\frac{1}{2}\left(\sigma^{-1}\right)_{ab}L^aL^b,\quad \dot{L}^i=\{L^i,H\} \tag{58}$$

In order to produce a CMS based on the Hamiltonian model (58) of course physical friction may be added, for instance considering the rotation energy dissipation due to the μSTDoF of a viscous fluid through which the rigid body is rotating. However, another way is possible, presented originally in [6] as far as the Author knows, and hence referred to as *Morrison's rotator*, from the name of the mathematical physicist who published [6]; indeed, any function $W(L^2)$, being $L^2\equiv\vec{L}\cdot\vec{L}$, is a Casimir of the Poisson bracket (57), hence any function

$$F(\vec{L}) = H(\vec{L}) + \alpha W(L^2) \tag{59}$$

may be used as free energy to build up an MBA, once the suitable semi-metric tensor G_{ij}, and a consequent semi-metric bracket $(.,.)$ are constructed:

$$G_{ij}(\vec{L}) = \frac{1}{\alpha}\left(\left| \partial_{\vec{L}} H \right|^2 \delta_{ij} - \frac{\partial H}{\partial L^i}\frac{\partial H}{\partial L^j} \right), \quad (f,g) = G^{ij}\frac{\partial f}{\partial L^i}\frac{\partial g}{\partial L^j}. \tag{60}$$

In (60) the projector orthogonal to the $\vec{L}$-gradient of H has been used to construct G, so it's blatant that $(f,H) = 0$ for any f. All in all, the right brackets to prepare a MBA with (59) as the free energy are ready:

$$\langle\langle f,g \rangle\rangle = \{f,g\} + (f,g), \quad \dot{f}(\vec{L}) = \langle\langle f(\vec{L}), F(\vec{L}) \rangle\rangle \quad \forall \quad f \in C^\infty(\mathbf{R}^3, \mathbf{R}). \tag{61}$$

It may be shown that the steady points of the system with free energy (59) are the configurations with $\vec{L}$ aligned along one principal axis of inertia, i.e. has only one component; THE CMS of dynamics (61) represents a free rigid rotator that relaxes getting aligned with one of its axes of inertia. Again, as it was already happening for the kinetic theory in § 3.5, there exist no macroscopic degrees of freedom the ordered energy of which is drained by the disordered microscopic degrees of freedom giving friction: rather, (61) represents an *algebrized dynamical system* with a time "time-reversible" symplectic part and an "ageing" metric part. It is pretty clear that this MBA works in the same way of a CMS with friction; of course, the delicate point is to understand that the function $W(L^2)$ of (59) has to do physically with the entropies seen until now.

4. Conclusions

After having gone through what presented here, I would like the reader to retain few important facts, listed here.

1. Classical dynamical systems with dissipation may be recast in an algebraic way, through a generalization of the symplectic brackets of Hamiltonian systems: this is the metriplectic bracket formalism. In bracket algebraic formalism the symmetry properties are much more under control and may be exploited.
2. The bracket algebra associated to a classical dissipative complete system is composed by the Poisson brackets and the Hamiltonian, that describe "what the system is made like" (i.e., what are its degrees of freedom and dynamical variables, what are its own fundamental time scales, the hard core of its phase space structure), and the metric brackets and the entropy, that describe "what the system ages like", relaxing to an asymptotically stable state. This kind of representation is referred to as *complete metriplectic system* (CMS).
3. The metric component giving rise to dissipation is crucially *symmetric and positive (semi-) definite*: this fact does allow irreversible motion to take place.
4. CMS may have their dissipative component either originated from the interaction of a "macroscopic" Hamiltonian system with microscopic degrees of freedom (usually treated statistically), giving rise to "friction" (the examples in § 3.3 and § 3.4); or from a "postulated" non-Hamiltonian interaction among dynamical variables already involved in the Hamiltonian component (the examples in § 3.5 and § 3.6).

5. In the cases studied in § 3.3 and § 3.4 (dissipation with friction) and in § 3.5 (dissipation due to particle collisions), the observable responsible for the non-Hamiltonian, metric part of the dynamical algebra is easily interpreted as the entropy of the system. In the system discussed in § 3.6 the interpretation of the same quantity is not completely clear.

As demonstrated during this work, metriplectic formalism promises to put together dynamical system theory and bracket algebra also in the presence of dissipation, involving an observable behaving as a Lyapunov function in the role of generator of the dissipative component. The Author is sure of the fact that, prolonging enough this path of research, the role of dissipation in fundamental Physics, even at elementary particle level, will be clarified; moreover, the algebraic nature of CMS is a blatant invitation to match what we learn at a classical level with the quantum world.

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Author Contributions

Besides putting together the whole material for the lecture, the Author contributed in the branch of research discussed here with the publications by him appearing in the References.

Conflicts of Interest

No conflict of interest has to be stated for the Author Massimo Materassi.

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