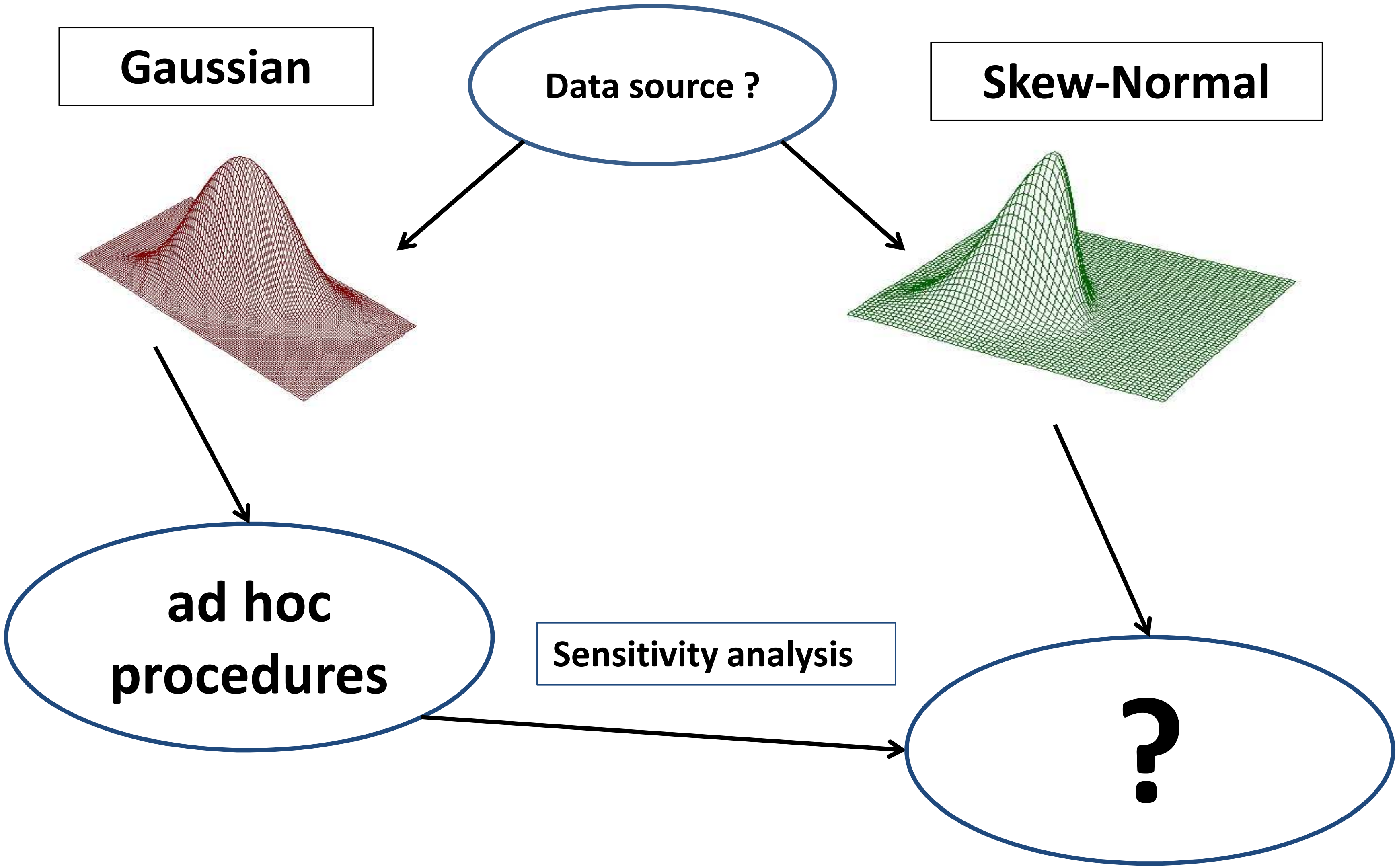
Local effect of asymmetry deviations from Gaussianity using information-based measures

Paloma Main, Universidad Complutense de Madrid, Spain
Jorge M Arevalillo and Hilario Navarro, UNED, Spain

INTRODUCTION



THE MULTIVARIATE SKEW-NORMAL

$$\boldsymbol{Z} \sim SN_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$$

location vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)'$ and scale matrix $\boldsymbol{\Omega}$

$$f(\boldsymbol{z}; \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\Omega}) = 2\phi_p(\boldsymbol{z} - \boldsymbol{\xi}; \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}'\boldsymbol{\omega}^{-1}(\boldsymbol{z} - \boldsymbol{\xi})) \quad : \quad \boldsymbol{z} \in \mathbb{R}^p$$

$\phi_p(\cdot\,; \boldsymbol{\Omega})$ p-dimensional Gaussian density with zero mean and full rank covariance matrix $\boldsymbol{\Omega}$

$\boldsymbol{\Phi}$ Distribution function of a standard Gaussian random variable

$\boldsymbol{w} = \text{diag}(w_1, \ldots, w_p)$ is a scale diagonal matrix with non negative entries such that

$$\bar{\boldsymbol{\Omega}} = \boldsymbol{w}^{-1}\, \boldsymbol{\Omega}\, \boldsymbol{w}^{-1}$$

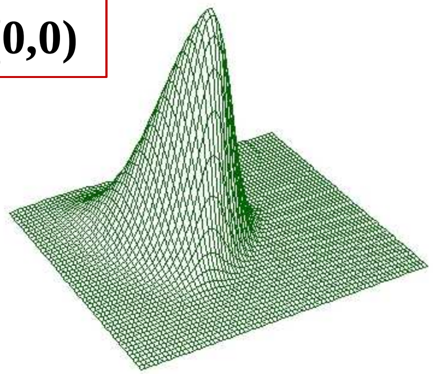
is a correlation matrix

$\boldsymbol{\alpha}$ is a p-dimensional shape vector regulating skewness

GRAPHICS

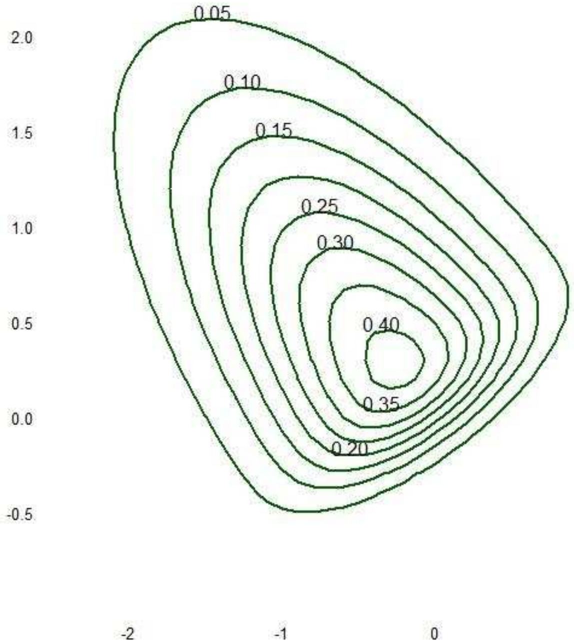
p = 2

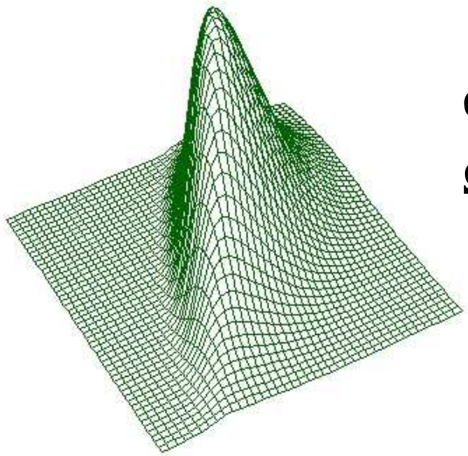
ξ = (0,0)



$\alpha = (5,-3)$

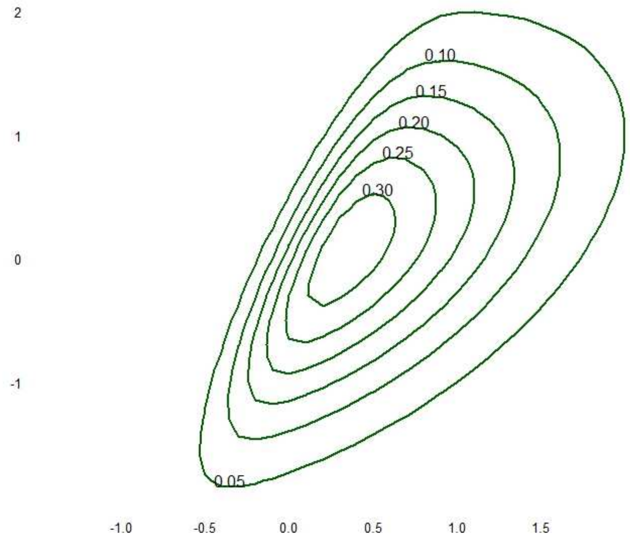
$\Omega = \begin{pmatrix} 1 & -0.7 \\ -0.7 & 1 \end{pmatrix}$





$\alpha = (-2,6)$

$\Omega = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$



MARDIA'S MEASURES OF SKEWNESS AND KURTOSIS FOR SN DISTRIBUTIONS

$$\gamma_{1,p}^{M} = 2(4-\pi)^2 \left\{ \frac{\boldsymbol{\alpha}'\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}}{\pi + (\pi - 2)\boldsymbol{\alpha}'\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}} \right\}^3$$

$$\gamma_{2,p}^{M} = p(p+2) + 8(\pi - 3) \left\{ \frac{\boldsymbol{\alpha}'\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}}{\pi + (\pi - 2)\boldsymbol{\alpha}'\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}} \right\}^2$$

$\boldsymbol{\alpha}'\bar{\boldsymbol{\Omega}}\boldsymbol{\alpha}$ scalar that summarizes the departure from normality

LOCAL DEVIATION INDICES

$\boldsymbol{Z} \sim N_p(\boldsymbol{0}, \boldsymbol{\Omega})$ and $\boldsymbol{Z}^{\boldsymbol{\alpha}} \sim SN_p(\boldsymbol{0}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$

Gaussian **Skew-Normal**

KULLBACK-LEIBLER DIVERGENCE MEASURE

$$D_{KL}(\mathbf{Z}, \mathbf{Z}^{\alpha}) = -E\left[\log 2\Phi\left(\boldsymbol{\alpha}'\boldsymbol{\omega}^{-1}\mathbf{Z}\right)\right]$$

INDICES

Local deviation index (LD): For every non-null direction **e** such that **e'e** = 1 consider, for ε > 0

$$\mathbf{Z}^{\varepsilon} \sim SN_p(\mathbf{0}, \mathbf{\Omega}, \varepsilon\, \mathrm{e})$$

then, after some calculations

$$LD(\mathrm{e}, \mathbf{\Omega}) = \lim_{\varepsilon \downarrow 0} \frac{D_{KL}(\mathbf{Z}, \mathbf{Z}^{\varepsilon})}{\varepsilon^2} = \frac{\mathrm{e}'\bar{\mathbf{\Omega}}\mathrm{e}}{\pi} \leq \frac{\mathrm{e}_1'\bar{\mathbf{\Omega}}\mathrm{e}_1}{\pi}$$

normalized eigenvector corresponding to the largest eigenvalue of $\bar{\Omega}$

EXAMPLE: PERMUTATION SYMMETRIC NORMAL

$$\mathbf{\Omega} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sigma^2 & \cdots & \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

with $\rho > \frac{1}{p-1}$

then

$$LD(\mathrm{e}, \mathbf{\Omega}) = \frac{\mathrm{e}'\bar{\mathbf{\Omega}}\mathrm{e}}{\pi} \leq \frac{\mathrm{e}_1'\bar{\mathbf{\Omega}}\mathrm{e}_1}{\pi} = \frac{1 + (p-1)\rho}{\pi}$$

DEVIATION INDICES

Relative local deviation index (RLD): For every non-null direction **e**

such that **e'e** = 1

$$RLD(\mathrm{e}, \mathbf{\Omega}) = \frac{\mathrm{e}'\bar{\mathbf{\Omega}}\mathrm{e}}{\mathrm{e}_1'\bar{\mathbf{\Omega}}\mathrm{e}_1}$$

normalized eigenvector corresponding to the largest eigenvalue of $\bar{\Omega}$

EXAMPLE: PERMUTATION SYMMETRIC NORMAL

p=2

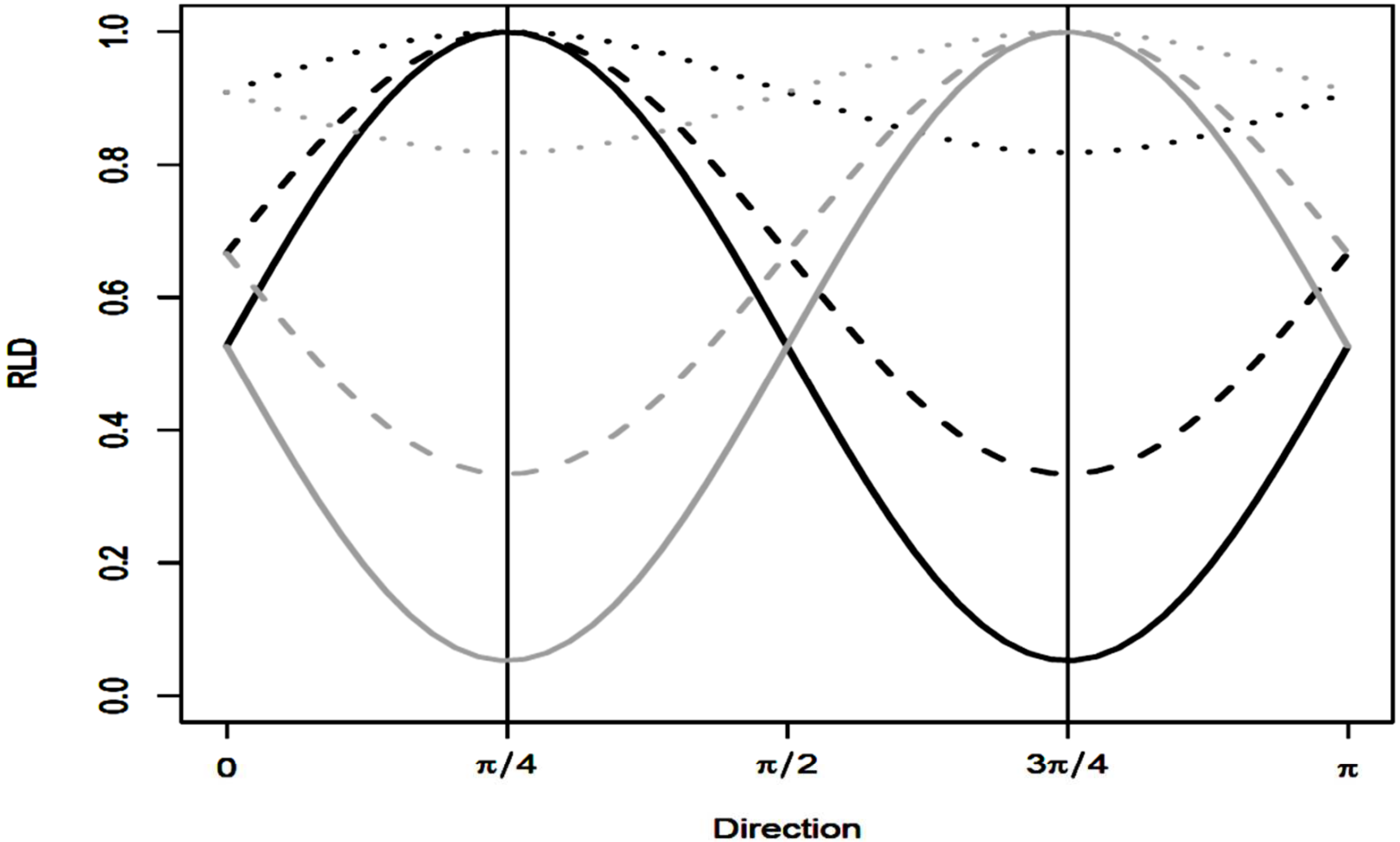


Figure 1: Plots of RLD against direction —angles $\theta \in [0, \pi]$— for different correlations: $\rho = 0.9$ (black solid curve), $\rho = 0.5$ (black dashed curve), $\rho = 0.1$ (black dotted curve) and $\rho = -0.9$ (gray solid curve), $\rho = -0.5$ (gray dashed curve), $\rho = -0.1$ (gray dotted curve).

RELATIVE CONDITIONAL SENSITIVITY

Let $\begin{pmatrix}\mathbf{Y}\\ \mathbf{X}\end{pmatrix} \begin{cases} k \\ p-k \end{cases}$ be a *p-dimensional* random variable

$\mathbf{Y} \mid \mathbf{X}$ *k-dimensional* conditional random variable

For densities $f_{(\mathbf{X},\mathbf{Y})}$ and $g_{(\mathbf{X},\mathbf{Y})}$ the "chain rule“

$$D_{KL}\left(f_{(\mathbf{Y},\mathbf{X})}, g_{(\mathbf{Y},\mathbf{X})}\right) = E_{\mathbf{X}}\left[D_{KL}\left(f_{\mathbf{Y}|\mathbf{X}}, g_{\mathbf{Y}|\mathbf{X}}\right)\right] + D_{KL}\left(f_{\mathbf{X}}, g_{\mathbf{X}}\right)$$

RELATIVE CONDITIONAL SENSITIVITY MEASURE (RCSM)

For every non-null direction **e** such that **e' e = 1**

$$\binom{\mathbf{Y}}{\mathbf{X}} \sim N_p(\mathbf{0}, \mathbf{\Omega}) \longrightarrow f_{(\mathbf{Y},\mathbf{X})}$$

$$\binom{\mathbf{Y}}{\mathbf{X}}^{\varepsilon} \sim SN_p(\mathbf{0}, \mathbf{\Omega}, \varepsilon\, \mathbf{e}) \;\; with\;\; \varepsilon > 0 \longrightarrow f^{\varepsilon}_{(\mathbf{Y},\mathbf{X})}$$

Definition

$$RCSM = \lim_{\varepsilon \downarrow 0} \frac{E_{\mathbf{X}}\left[D_{KL}\left(f_{\mathbf{Y}|\mathbf{X}}, f^{\varepsilon}_{\mathbf{Y}|\mathbf{X}}\right)\right]}{D_{KL}\left(f_{(\mathbf{Y},\mathbf{X})}, f^{\varepsilon}_{(\mathbf{Y},\mathbf{X})}\right)} = 1 - \lim_{\varepsilon \downarrow 0} \frac{D_{KL}\left(f_{\mathbf{X}}, f^{\varepsilon}_{\mathbf{X}}\right)}{D_{KL}\left(f_{(\mathbf{Y},\mathbf{X})}, f^{\varepsilon}_{(\mathbf{Y},\mathbf{X})}\right)},$$

RESULTS

With zero mean and partitioning $\mathbf{\Omega}$, $\boldsymbol{\omega}$ and $\mathbf{e}$

$$\Omega = \begin{pmatrix} \Omega_{\mathrm{YY}} & \Omega_{\mathrm{YX}} \\ \Omega_{\mathrm{XY}} & \Omega_{\mathrm{XX}} \end{pmatrix} , \bar{\Omega} = \begin{pmatrix} \bar{\Omega}_{\mathrm{YY}} & \bar{\Omega}_{\mathrm{YX}} \\ \bar{\Omega}_{\mathrm{XY}} & \bar{\Omega}_{\mathrm{XX}} \end{pmatrix} \quad \omega = \begin{pmatrix} \omega_{\mathrm{Y}} & 0 \\ 0 & \omega_{\mathrm{X}} \end{pmatrix} \quad \mathrm{e} = \begin{pmatrix} \mathrm{e_Y} \\ \mathrm{e_X} \end{pmatrix}$$

$$RCSM = \frac{\mathrm{e}'_{\mathrm{Y}}\bar{\Omega}_{\mathrm{YY.X}}\mathrm{e_Y}}{\mathrm{e}'\bar{\Omega}\mathrm{e}}$$

with

$$\bar{\Omega}_{\mathrm{YY.X}} = \bar{\Omega}_{\mathrm{YY}} - \bar{\Omega}_{\mathrm{YX}}\bar{\Omega}_{\mathrm{XX}}^{-1}\bar{\Omega}_{\mathrm{XY}}$$

the Schur complement of the submatrix $\bar{\Omega}_{\mathrm{XX}}$ in $\bar{\Omega}$

EXTREME IMPACTS OF NON-GAUSSIANITY

Directions through which the impact of non-Gaussianity attains a maximum or a minimum

as a generalized Rayleigh's quotient

$$RCSM = \frac{\mathbf{e}'\bar{\Omega}^*_{\mathbf{YY.X}}\mathbf{e}}{\mathbf{e}'\bar{\Omega}\mathbf{e}}$$

with

$$\bar{\Omega}^*_{\mathbf{YY.X}} = \begin{pmatrix} \bar{\Omega}_{\mathbf{YY.X}} & \mathbf{0}_{k\times(p-k)} \\ \mathbf{0}_{(p-k)\times k} & \mathbf{0}_{(p-k)\times(p-k)} \end{pmatrix}$$ an extended version of $\bar{\Omega}_{\mathbf{YY.X}}$

Solution: the maximum and minimum eigenvalues of

$$\bar{\Omega}^{-1}\bar{\Omega}^*_{\mathbf{YY.X}}.$$

EXAMPLE: PERMUTATION SYMMETRIC NORMAL

$$\bar{\Omega}^{-1}\bar{\Omega}^{*}_{\mathbf{YY.X}} = \begin{pmatrix} \mathbf{I}_k & \mathbf{0}_{k\times(p-k)} \\ -\frac{\rho}{1+\rho(p-k-1)}\mathbf{1}_{(p-k)\times k} & \mathbf{0}_{(p-k)\times(p-k)} \end{pmatrix}$$

direction of asymmetry with maximum local effect on the conditional distribution

$$\mathbf{e}_{\max} = \begin{pmatrix} \mathbf{e_Y} \\ \mathbf{e_X} \end{pmatrix} \quad \left\{ \begin{array}{l} \mathbf{e_Y} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k\times 1}, \\ \mathbf{e_X} = -\dfrac{\rho}{1+\rho\,(p-k-1)}\mathbf{1}_{(p-k)\times 1} \end{array} \right.$$

EXAMPLE: BIDIMENSIONAL CASE

Maximum impact $\longrightarrow$ normalized eigenvector associated with the highest eigenvalue λ_1**=1**

$$\mathbf{e}_{\max} = \begin{pmatrix} 1/\sqrt{1+\rho^2} \\ -\rho/\sqrt{1+\rho^2} \end{pmatrix}$$

Minimum impact $\longrightarrow$ Maximum impact on the marginal univariate conditioning distribution

$$\mathbf{e}_{\min} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

EXAMPLE: GRAPHICS

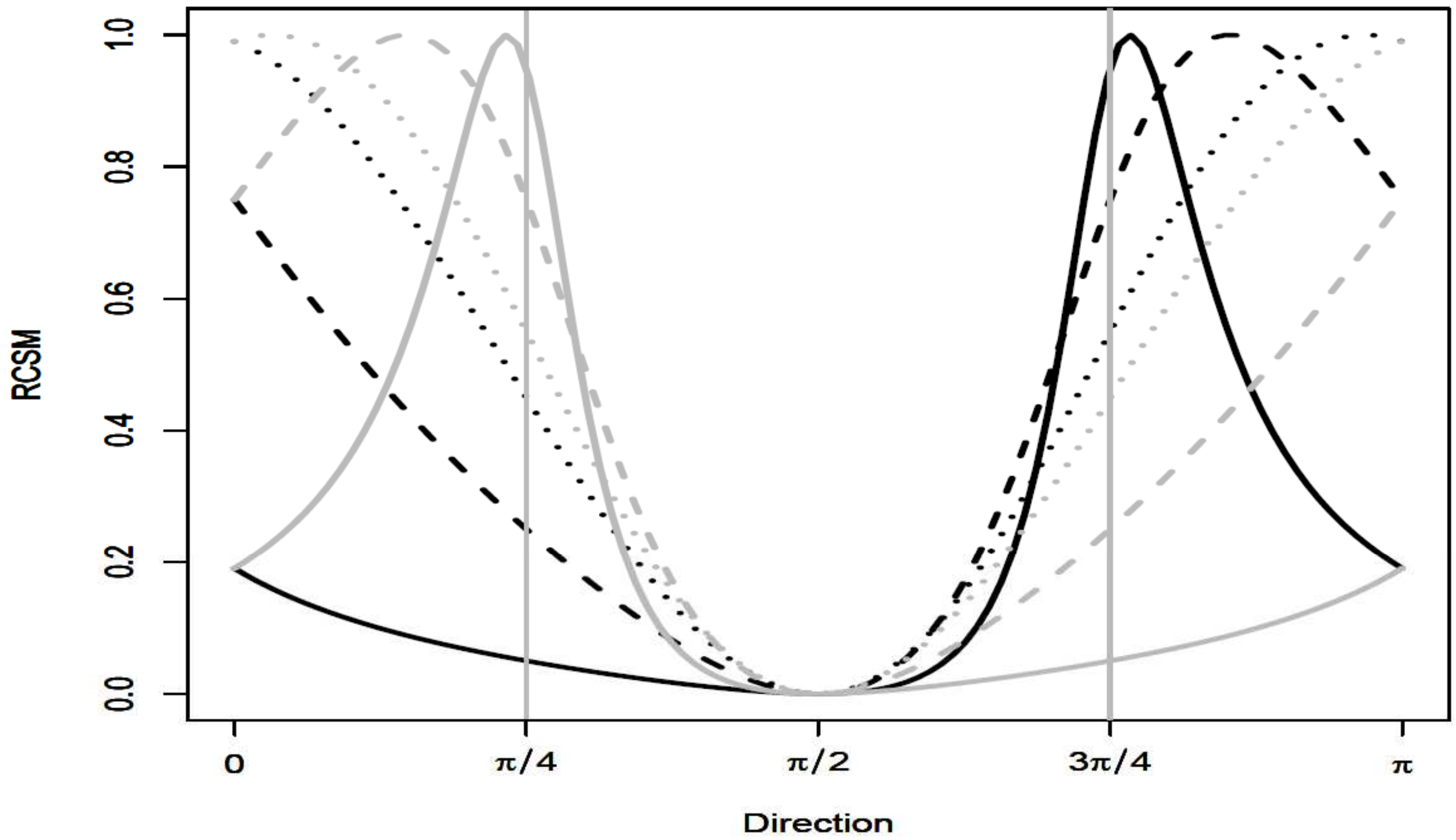


Figure 2: Plots of RCSM against direction —angles $\theta \in [0, \pi]$— for different correlations: $\rho = 0.9$ (black solid curve), $\rho = 0.5$ (black dashed curve), $\rho = 0.1$ (black dotted curve) and $\rho = -0.9$ (gray solid curve), $\rho = -0.5$ (gray dashed curve), $\rho = -0.1$ (gray dotted curve).

CONCLUSIONS

- The problem of departures from Gaussianty due to asymmetry is studied
- The asymmetry is regulated by the family of Skew-Normal (SN) distributions
- Local deviation **(LD)** and relative local deviation **(RLD)** indices are introduced
- Mardia and Malkovich-Afifi's measures of skewness and LD index are related
- Also, the effect of asymmetry in the conditional distributions is analyzed, proposing a relative conditional measure **(RCSM)** to evaluate perturbation effects
- Finally, the directions of asymmetry for which slight perturbations lead to extreme impact on the conditional distribution are determined