proceedings

MDPI

Proceedings

Cuts, Qubits, and Information †

Rossella Lupacchini [1,*]

[1] Department of Philosophy and Communication Studies, University of Bologna, Italy (via Zamboni 38, I-40126 Bologna)

* rossella.lupacchini@unibo.it

† Presented at the IS4SI 2017 Summit DIGITALISATION FOR A SUSTAINABLE SOCIETY, Gothenburg, Sweden, 12-16 June 2017.

Published: date: 9 June 2017

Abstract: In his search for the 'essence' of continuity, Richard Dedekind (1872) discovered the notion of cut. Epistemologically speaking, a cut produces a separation of a simply infinite system into two parts (*Stücke*) such that all the elements of one part are screened off all the elements of the other. The distinct continuity of a two-state quantum system is encapsulated in the notion of *qubit*, the basic 'unit' of quantum information. A qubit secures an infinite amount of information, which, however, appears to be only penetrable through 'sections' of classical bits. Whereas Dedekind's cuts dwell on the discrete of number theory, the theory of nature is primarily concerned with continuous transformations. In contrast with Dedekind's line of thought, could the notion of information be derived from a 'principle' of continuity?

Keywords: continuity, cuts, distinguishability, correlations

---

1. The "Phenomenon of the Cut"

Dedekind's main concern was to clean the science of numbers from foreign notions, such as measurable quantities or geometrical evidence. Hence, the real challenge was to extract a purely arithmetic and perfectly rigorous definition of the essence of continuity from the discrete of rational numbers.

The *vexata quaestio* of "continuity and irrational numbers" originated with the Pythagorean discovery of incommensurable ratios. In the eyes of Pythagoreans, however, it was the divergence between the harmony of geometrical forms and the "atomism" of numbers to be disturbing. The early Pythagoreans "did not really distinguish numbers from geometrical dots. Geometrically, then, a number was an extended point or a very small sphere" [1] (p. 29). By contrast, in Dedekind's view, "numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things" [2] (p. 791).

Amazingly, Dedekind extracted the essence of continuity from cuts. Considering that every point produces a separation of the straight line into two parts such that every point of one part lies to the left of every point of the other, Dedekind recognized the special character of continuity in the converse, i.e. in the following principle:

> If all points of the straight line fall into two classes such that every point of the first class (*Klasse*) [A_1] lies to the left of every point of the second class [A_2], then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. [3] (p. 771)

What is precisely determined is primarily the *division* itself.[1] Hence, whenever we have a cut (A_1, A_2) produced by no rational number, we can create a new number, an *irrational* number, which we regard as completely defined by this cut (cf. [3] p. 773). So the system of real numbers is obtained by filling up the gaps in the domain of rational numbers and making it continuous; "*taking the object that fills each gap to be essentially the gap itself*" [4] (p. 23).

Dedekind's notion of 'cut' raises distinguishability on to a higher level of generality – from (integer) numbers to classes, from elements to properties – taking into account not solely the relations of one individual number to another, but also the relations between (infinite) sets of elements. As Dedekind emphasized, if "one regards the irrational number as the ratio of two measurable quantities," then this manner of determining it is already set forth in the clearest possible way by Euclid. However, a presentation "in which *the phenomenon of the cut in its logical purity* is not even mentioned, has no similarity whatever to mine, inasmuch as it resorts at once to the existence of a measurable quantity, a notion which (...) I wholly reject" [2] (p. 794).

2. From Atoms to Qubits

While Dedekind's axiom of continuity guarantees the logical purity of real numbers, physics needs measurable quantities to unravel the continuity of nature into elements. It is noteworthy that both the Pythagorean arithmetical atomism and the Democritean physical atomism were stuck on 'continuity.' Since the harmony of (natural) forms ought to be expressed by (whole) numbers, there was no way to fill the gap between the finite and the infinite. If the discovery of incommensurable ratios meant the departure of geometric constructions from arithmetic operations, Zeno's paradoxes made it clear that *motion* is not attainable by summing up an infinite series of discrete states.

It is a great achievement of quantum theory to have read the divide between measurable quantities and continuous transformations as a dialectic contrast and to have made it the source of physical meaning.

2.1. Ghost fields

Interestingly, a '*quantum* Zeno effect' was defined by John von Neumann [5]: a sequence of measurements frequently performed on a quantum system can slow down or even halt the evolution of the state. As a consequence of the quantum Zeno effect, in a classical interference experiment,[2] when a photon emerging from a Mach-Zehnder interferometer informs that a 'which-path' measurement was set on its way, the probability that no measurement was actually performed (i.e., no photon-observer interaction took place) could be stretched to the limit of 1.

The debate on the impact of 'null-result' measurements on the behaviour of quantum systems or, more generally, on the nature of *quantum interference* urged the search for a more 'sensible' description of physical reality. It is well known that Einstein, Podolsky, Rosen's celebrated essay [6] was supposed to highlight the conflict between the completeness of the quantum physical description of physical reality and Heisenberg's uncertainty relations, but in fact it drew attention to a form of 'non-locality' underlying quantum systems. When measurements are performed on certain pairs of particles, the values of the same physical quantity for the two separated particles appear *instantly correlated*. Seemingly, the failing attempts to find a reasonably 'realistic' explanation of quantum interference effects (*via* experiment) led Einstein to coin the term "ghost fields" (*Gespensterfelder*) for quantum waves.

2.1. Perspectives on distinguishability

---

[1] Any separation of the domain of rational numbers into two classes, A_1 and A_2 , such that every number of one class is less than every number of the other, defines a *real* number.

2 Think of photons set going through a Mach-Zehnder interferometer. After encountering the first beam splitter each photon can choose between two mutually exclusive paths to reach the second beam splitter.

Rather than questioning *non-locality* quantum correlations enlighten a notion of *non-separability*, called 'entanglement.' As Schrödinger [7] observed, two quantum systems interact in a way such that only the properties of the pair are defined. Consider for instance the spin components. Although any individual particle holds a set of well-defined values, once two particles get entangled in a pair, the spin of one particle and that of the other go in the *same* direction or in *opposite* directions; 'being the same' and 'being opposed' are *not* values of physical quantities, but *relations* between states. Accordingly, quantum theory brings about pure 'relational properties' that do not work for absolute states of a system.

As for measurement of a system's state, quantum theory also unveils its essential relational character. No measurement makes sense (to anyone) out of the relation between two 'subjects': the system and the observer. Any physical system numbers a set of characteristic 'potential' features. To become 'temporarily' real, any of these features is bound to a feasible system-observer interaction. In this perspective, any measurement filters out information as a value of a 'pair-dependent' physical quantity (cf. Rovelli [8]). To the extent that measurement is taken as an interaction where the perspective chosen (observer) determines the range of values of the physical quantity (observable), it demands for sharpening the very notion of *distinguishability*. This leads quantum theory to introduce *complex probability amplitudes,* which measure the angular separation between distinct values from distinct perspectives and must be squared to obtain probabilities.

Thus, the classical tenet that the value of a physical quantity is not affected by measurement must be revised. It is wrong to attribute a feature to a quantum system until a measurement has brought it to a close by an act of irreversible amplification (cf. Wheeler [9]).

2.3. The "elementary quantum phenomenon"

"One who comes from an older time and is accustomed to the picture of the universe as a machine built out of 'atoms' is not only baffled but put off when he reads [...] Leibniz's conception of the ultimate building unit, the monad" [9] (p. 560). What Leibniz wrote about the "monad", Wheeler observed, is more relevant to what he called "quantum phenomenon" than to anything one has ever called an 'atom'. The very word "phenomenon", as Wheeler stressed, is the result of a long lasting debate between Bohr and Einstein about the logical self-consistency of quantum theory and its implications for *reality*: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon."

But Leibniz's monad has neither extension, nor shape, hence it is *not* observable. It is *a simple substance and a unity of perceptions*. As a unity of perceptions, it contains the whole universe. As a simple substance, it is not a 'tangible' thing, but rather the 'perceiving faculty' itself. Indeed perception plays a key role in Leibniz's *Monadology,* for it performs the inner constant change, and also, as a function of correlation, enables monads to express each other:

> This *interconnection* or accommodation of all created things to each other, and each to all the others, brings it about that each simple substance has relations that express all the others, and consequently, that each simple substance is a perpetual *living mirror* of the universe. (*Monadology* 56, [10])

How to draw 'actual' perceptions – *i.e.* natural phenomena – from an impenetrable faculty of perceiving, from the infinite unity of perceptions encapsulated in each monad? More than to the quantum phenomenon, what Leibniz wrote about the monad seems to apply to the *qubit*.

Like a monad, a qubit, which is the basic unit of quantum information theory, involves an infinite multitude of states (perceptions). As a two-state quantum system, it can be described as a coherent superposition of two mutually exclusive states. It follows that there is no way to extract information from qubits other than by measuring them with 'yes-no' questions.

3. The Essence of Information

In the 'artificial' construction of a theoretical model – be it the Euclidean geometry or the universal computer – one starts with distinct elements and ponders how to achieve the connecting structure. In the attempt to figure out the intelligence of nature, one starts with the structure and tries to analyze it into elements; at its heart, the wish to grasp the ultimate 'principle of existence': a principle of *metamorphosis*.

Reversing the Euclidean perspective, in his search for a general *geometric characteristic*, Leibniz pursued the 'inner principle' of geometry:

> Imagine taking two points in space, hence conceiving the indeterminate straight line through them; one thing is that each point is regarded individually as single, another thing is that both are regarded as simultaneously existing; besides the two points, something else is needed for seeing them as co-existent in their respective positions. When we consider one of the two points as if we took its position and looked at the other (point), what the mind determines is called *direction*. [11] (p. 278)

Time enters geometry and generates the concept of space: "Space is the continuity in the 'order of co-existence' according to which, given the co-existence relation in the present and the law of changes (*lege mutationis*), the co-existence relation in any given time can be defined."

For Leibniz, the whole universe is contained in every monad from the very beginning, and the simple substance of monad coincides with the continuity principle of the disclosure of itself. Therefore, every monad must be also endowed with an original faculty of *representing*, which makes it able to match the variety of phenomena through perceptions. To deliver 'information' about the universe, however, perceptions must become *observable* in the guise of phenomena.

Now, like a direction, each perception needs two *co-existent elements* to be determined by an external observer. Thus, all measurable quantities (*i.e.*, the basic constituents of physics) must come into 'existence' as *pairs*. This imposes one constraint on natural phenomena: given the infinity of perceptions, the number of natural elements must be the logarithm to the base two of that infinity. In this sense, Pythagoras correctly drew the geometry of nature from whole numbers. On the other hand, Leibniz insightfully saw the infinite multitude of natural forms as *related* to the different points of view of each monad.

In Leibniz's world, however, there is no conflict between the continuity of the simple substance and the distinguishability of perceptions, because each monad is a "*living mirror* of the universe." By contrast, in the (quantum) physical world, the spectral decomposition of any physical quantity depends on the perspective chosen. The *substance* of nature is captured by a unitary transformation, but physical *knowledge* is born of distinguishability. Shifting the attention from the notion of 'state' of a system to "the notion of information that a system has about another system" [8], quantum theory also focuses upon the notion of distinguishability as a 'cross-ratios' between values and perspectives. Thus, the essence of information springs from *correlations*.

References

1. Kline, M. *Mathematical Thought from Ancient to Modern Times*. Oxford Univ. Press: Oxford, 1972; Volume 1.
2. Dedekind, R. Was sind und was sollen die Zahlen (1888). In *From Kant to Hilbert: A source book in the foundations of mathematics.* Ewald, W., Ed.; Oxford Univ. Press: Oxford, 1996; Volume 2, pp. 790-833.
3. Dedekind, R. Continuity and Irrational Numbers (1872). In *From Kant to Hilbert*; *cit.* pp. 765-779.
4. Stillwell, J. *Roads to Infinity*. A. K. Peters: Natick, MA, 2010.
5. von Neumann, J. *Mathematische Grundlagen der Quantenmechanik*. Springer: Berlin, 1932.
6. Einstein, A.; Podolski, B.; Rosen, N. Can Quantum Mechanical Description of Physical Reality Be Considered Complete? *Physical Review* 1935, **47**: 777-780.
7. Schördinger, E. Die gegenwrtige Situation in der Quantenmechanik, *Naturwissenschaften* 1935, **23**: 807-812.
8. Rovelli, C. Relational Quantum Mechanics. *Int. J. Theor. Phys*. 1996, 35: 1637-1678.
9. Wheeler, J. A. The Computer and the Universe, *Int. J. Theor. Phys*. 1982, **21**: 557-572
10. Leibniz, G. W. *Philosophical Essays*. Ariew, R., Garber, D., Eds.; Hackett Publishing Company: Indianapolis & Cambridge, 1989.
11. Leibniz, G. W. *La caractétistique géométrique*. Echeverría, J., Parmentier, M., Eds.; Librairie philosophique J. Vrin: Paris, 1995.