EXPONENTIAL OR POWER LAW - HOW TO SELECT A STABLE DISTRIBUTION OF PROBABILITY IN A A PHYSICAL SYSTEM

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Boltzmann’s exponential and Gibbs’ thermodynamics

Gibbs' entropy (normalized to k_B)

$$S = -\sum_k p_k \ln p_k$$

Once properly maximized, leads to Boltzmann's exponential distribution of microstates in canonical systems (generalized to grand-canonical…)

$$p_k \propto e^{-\beta \varepsilon_k}$$

The probability of a fluctuation of an arbitrary parameter λ around a S = max state follows Einstein's formula

$$w(\lambda) \propto e^{\Delta S(\lambda)}$$

and the variance of such fluctuation is

$$\left(\frac{\partial^2 S}{\partial \lambda^2}\right)^{-1}$$

Gibbs' entropy is additive: if A', A'' are independent systems then

$$S(A' + A'') = S(A') + S(A'')$$

Then, it can be written as the sum of the Gibbs' entropies of all small mass elements the system is made of $\rightarrow$ a local entropy density s exists (ρ mass density)

$$S = \int \rho s \, dV$$

If Local Thermodynamical Equilibrium (LTE): ($\rightarrow$ Gibbs-Duhem)

$$s = max$$

If LTE holds everywhere and at all times: General Evolution Criterion (GEC)

$$\frac{dT^{-1}}{dt}\frac{d\left(\rho u\right)}{dt}-\rho\sum_{h}\frac{d\left(\mu_{h}^{o}T^{-1}\right)}{dt}\frac{dc_{h}}{dt}-\left[\rho^{-1}T^{-1}\frac{dp}{dt}+\left(u+\rho^{-1}p\right)\frac{dT^{-1}}{dt}\right]\frac{d\rho}{dt}\leq 0$$

If $t \to \infty$ and the system relaxes to a stable, steady (*relaxed*) state with Boltzmann exponential distribution of microstates in all small mass elements at all times, then GEC rules relaxation regardless of detailed model

Glansdorff et al. 1964, Di Vita 2010

q-exponential and Tsallis' thermodynamics

Tsallis' entropy (normalized to k_B)

$$S_q = -\sum_k (p_k)^q \ln_q p_k$$

Tsallis 1988, Tsallis et al. 1998

$$\lim_{q\to 1} S_q = S$$

$$\lim_{q\to 1} ln_q(x) = ln\,(x)$$

Once properly maximized, leads to
q-exponential distribution of microstates
In canonical systems (generalized to grand-canonical…) $\rightarrow$ power law with exponent $1/(1-q)$

$$p_k \propto {exp_q}^{-\beta\varepsilon_k}$$

$$\lim_{q\to 1} exp_q(x) = exp\,(x)$$

What about q ?

Tsallis' entropy is nonadditive:

$$S_q(A' + A'') = S_q(A') + S_q(A'') + (1 - q)S_q(A')S_q(A'')$$

Then, it can not be written as the sum of the Gibbs' entropies of all small mass elements the system is made of
$\rightarrow$ no local entropy density s exists

$\rightarrow$ NO LTE $\rightarrow$ NO GEC $\rightarrow$ only model-dependent info
on relaxation (as $t \rightarrow \infty$) with power-law distribution of microstates

What about q ?

NLFP

E.g.: q-dependent, possibly nonlinear, 1D Fokker-Planck (NLFP) equation for continuous distribution function $P(x, t)$, where a force $A = A(x)$ is counteracted by a diffusion process, represented by a diffusion coefficient D

$$\frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad ; \quad J = \frac{1}{\eta}\left(AP - Dq' P^{q'-1} \frac{\partial P}{\partial x}\right) \qquad A = A(x) \qquad q' \equiv 2 - q$$

Casas et al. 2012, Wedemann et al. 2016

What about q ?

NFLP for $J = 0$

$\rightarrow$ Steady state solution is just the q-exponential: $P_{J=0,q} \propto \exp_q \left(\int^x dx' \frac{A}{D} \right)$

$\rightarrow$ $S_{q'} = \int dx \frac{P - P^{q'}}{q' - 1}$ is max. when the solution is $P_{J=0,q}$

Haubold et al. 2004, Ribeiro et al. 2012, Wedemann et al. 2016

What about q ?

NFLP for $J \neq 0$

$\rightarrow$ An H-theorem holds (provided that $A(x)$ is well-behaved at ∞)

$\rightarrow$ Relaxation does occur!

Casas et al. 2012

What about q ?

NFLP for $J \neq 0$

$\rightarrow$ If $J \neq 0$ then: $\Pi_{q'} - \frac{dS_{q'}}{dt} = \frac{1}{D} \int dx A J$ $\Pi_{q'} = \frac{\eta}{D} \int dx \frac{|J|^2}{P}$

Casas et al. 2012

$\Pi_{q'}$ is the amount (> 0) of $S_{q'}$ which is produced inside the bulk of the system in a time interval dt .

$\frac{1}{D} \int dx A J$ is the amount of $S_{q'}$ which is exchanged with the external world in dt .

$\rightarrow$ J represents the interaction with the external world;
in isolated system $J = 0$ (its value is a boundary condition on NLFP)

If a perturbation leaves the latter interaction unaffected then the increment $dS_{q'}$ of $S_{q'}$ in a time interval dt is $dS_{q'} = dt \cdot \Pi_{q'}$

What about q ?

Mapping Tsallis onto Gibbs

A monotonically increasing, additive function of S_q exists even for $q \neq 1$!!

Tsallis 1988,
Abe 2001,
Vives et al. 2002

$$\widehat{S}_q \equiv \frac{\ln\left(1+(1-q)\,S_q\right)}{1-q}$$

$$\widehat{S}_q\left(A' + A''\right) = \widehat{S}_q\left(A'\right) + \widehat{S}_q\left(A''\right)$$

$$\frac{d\widehat{S}_q}{dS_q} > 0$$

$$\lim_{q\to 1} \widehat{S}_q = S_{q=1}$$

These facts have a lot of consequences… $\rightarrow$

$\rightarrow \widehat{S_q} = max$ if and only if $S_q = max$

$\rightarrow$ Moreover: LTE, GEC formally unchanged provided that we replace S_q with $\widehat{S_q}$ ...
(**mapping** of Tsallis' onto Gibbs' thermodynamics)

$\rightarrow$ ... and relaxation behaves formally the same way regardless of q, in particular...

$\rightarrow$ ... the variance of fluctuations of λ around a $\widehat{S_q} = max$ state is $\left(\frac{\partial^2 \widehat{S_q}}{\partial \lambda^2}\right)^{-1}$ ...

$\rightarrow$ ... which implies (as $\frac{d\widehat{S_q}}{dS_q} > 0$) that the variance around a $S_q = max$ state is $\propto \left(\frac{\partial^2 S_q}{\partial \lambda^2}\right)^{-1}$

N.B. variance is always larger for Tsallis than for Gibbs! Vives et al. 2002

The quest for q:
NLFP …with J $\approx$ 0

In NLFP? q = const. However, nothing changes if $q = q(t)$ provided that $|d \ln q/dt| \gg |d \ln P/dt|$ (slow evolution)

$\rightarrow$ Slow evolution is a succession of relaxed states

$\rightarrow$ If $J \approx 0$ (i.e., the interaction with the external world is weak) then the relaxed state at time t corresponds just to $S_{q'=q'(t)} \approx max$ with $P \approx P_{J=0,q}$ ...

$\rightarrow$ ... and the variance of fluctuations of λ around a relaxed state is $\propto \left(\frac{\partial^2 S_{q'}}{\partial \lambda^2}\right)^{-1}$ ...

$\rightarrow$ ... which in a time interval dt is $\propto dt \cdot \left(\frac{\partial^2 \Pi_{q'}}{\partial \lambda^2}\right)^{-1}$

The larger the variance, the larger the fluctuations of λ which the relaxed state is stable against $\rightarrow$
the larger the variance, the more stable the relaxed state, the larger the fluctuations of λ which the probability distribution of the relaxed state is stable against

λ is arbitrary$\rightarrow$ we may take $d\lambda = dq'$, i.e. we deal with stability against (slow) fluctuations of the slope (depending on q') of the probability distribution

The most stable distribution function against fluctuations of q': $\frac{\partial^2 \Pi_{q'}}{\partial q'^2} = 0$

$\rightarrow$ This corresponds to an extremum of $\frac{\partial \Pi_{q'}}{\partial q'}$

$\rightarrow$ This is a minimum, as far as $J \approx 0$ at least. In the latter case, indeed:

$\rightarrow$ $dS_{q'} = dt \cdot d\,\Pi_{q'} = dt \cdot dq' \cdot \frac{d}{dq'} \Pi_{q'}$ s the amount of $S_{q'}$ produced in the time dt by the fluctuation dq'; it is ≥ 0 for $q' = 1$ (Gibbs' case!) as fluctuations involve irreversible physics and achieves its minimum value 0 at equilibrium (where $dS = 0$) of an isolated system (where J = 0).

$\rightarrow$ But GEC describes relaxation the same way regardless of q' and the structure of the relaxed state is modified only slightly for $J \approx 0$, hence $\frac{d}{dq'} \Pi_{q'}$ is still a minimum (even if non-zero), not a maximum!

Allowable range for $z = q' - 1 = 1 - q$: $0 \leq z < 1$ ($z = 0$ is Gibbs) Borland 1998

If $J \approx 0$, Taylor-series development of J in powers of z lead to the following useful formulas, which allow us to compute $\frac{d}{dz}\Pi_z$ once $A(x)$ and D are known:

$$\Pi_z = \Pi_{z=0} + \sum_{n=1}^{\infty} a_n z^n$$

$$a_n = \frac{(-1)^{n-1}(2J)}{(n-1)!} \int_0^{u_1} \mathrm{d}u A(u) \left[1 + \frac{1}{n}\left(\ln P_0 + \int_0^u \mathrm{d}u' A(u')\right)\right] \left[\ln P_0 + \int_0^u \mathrm{d}u' A(u')\right]^n$$

$$P_0 = \frac{1}{D \int_0^{u_1} \mathrm{d}u \exp\left[\int_0^u \mathrm{d}u' A(u')\right]}$$

$$\int_0^{u_1} \mathrm{d}u A(u) = 1$$

A rule for finding q in our NLFP!

If NLFP leads to a relaxed state (well-behaved $A(x)$) and $J \approx 0$ then the probability distribution of microstates in the relaxed state which is more stable against slow fluctuations of its own slope is the q-exponential with $q = 1 - z$ (similar to a power law with exponent z^{-1}) and z such that $\frac{d}{dz}\,\Pi_z$ = min. and that $0 < z < 1$.

In this case, power-law is stable against larger fluctuations than Boltzmann epoxnential, because the variance of the latter is always lower →

If such z does not exist, then if a relaxed state exists then its probability distribution is a Boltzmann's exponential .

N.B. Variance of fluctuations around a power-law distribution are always larger.

BUT… Why we have to depend on $J \approx 0$???

The quest for q: noisy 1D maps

Application: 1D, discrete, autonomous map

$$i = 0, 1, 2, \ldots$$

$$t' = i \cdot \Delta t'$$

$$Q_{i+1} = G\left(Q_i\right)$$

$$Q_i = \frac{x(t')}{\Delta t'}$$

The system evolves along a time interval $\gg \Delta t'$ $(i \to \infty)$

$$Q_{i+1} = \frac{x(t' + \Delta t')}{\Delta t'}$$

$\downarrow$

$Q_{i=0} = Q_0$ is known

$$\frac{dx(t')}{dt'} = A\left(x\right)$$

$$A\left(x\right) \equiv G\left(x\right) - x$$

Noise? Stochastic equation

$$t \equiv \eta \cdot t'$$

$$h(x,t) \propto P(x,t)^{\frac{z}{2}},\ 0 \leq z < 1$$

Borland 1998

$$\langle \zeta \rangle = 0$$

$$\langle \zeta(t)\, \zeta(t') \rangle = 2\eta D \delta(t - t')$$

$$\eta \frac{dx}{dt} = A(x) + h(x,t)\, \zeta(t) \quad \text{where} \quad A(x) \equiv G(x) - x$$

N.B. z unknown; noise may be either additive ($z = 0$) or multiplicative ;

$A(x)$ and D represent dynamics and noise level respectively;

$\eta > 0$ is arbitrary.

The stochastic equation is associated with NLFP (the probability distribution of the solution x of the stochastic equation is the solution P of NLFP):

$$\frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad ; \quad J = \frac{1}{\eta}\left(AP - Dq' P^{q'-1}\frac{\partial P}{\partial x}\right)$$

η is arbitrary $\rightarrow$ we choose it in such a way that the approximation $J \approx 0$ applies $\rightarrow$ we need no more justification of $J \approx 0$ and **our rules apply!**

Relaxed solutions of NLFP $\leftrightarrow$ the probability distributions for the noise affected Q_i as $i \rightarrow \infty \rightarrow$ **then…**

A rule for noise-affected maps!

Let a 1D, discrete, autonomus map $Q_{i+1} = G(Q_i)$ be affected by noise (no matter if additive or multiplicative) and let the Q_i's distribute as $i \to \infty$ along a probability distribution $P(Q_i)$. Then:

a) *If z exists such that $0 < z < 1$ and $\frac{d}{dz}\Pi_z$ = min then $P(Q_i)$ is a q-exponential with $q = 1 - z$ (similar to a power law with exponent z^{-1})*

b) *Otherwise, $P(Q_i)$ is a Boltzmann's exponential*

N.B. Variance of fluctuations around a power-law distribution are always larger.
N.B. Only info on dynamics ($A(x) = G(x) - x$) and level noise (D) required!!!

Theory vs. (numerical) exp.

Example: the map of Sànchez et al. 2007 $G(x) = rx \exp(-|1-a|x)$

Relevant to econophysics for $a = 0.8, 0 < r < 7$ ($x \geq 0$is richness, $P(x)$ its distribution). Noise applied to the initial condition (which gets randomized).
($x \geq 0 \rightarrow$ Boltzmann's exponential $\propto e^{-\beta x}$ = Gaussian (random) $\propto$ $e^{-\beta y^2}$in $y \equiv \sqrt{x}$)

Looking (with MATHCAD) for the minima of $\frac{d}{dz}\Pi_z$ in the interval $0 < z < 1$,

the easiest way is to look for zeroes of $\frac{d^2}{dz^2}\Pi_z$

which cross the zero line with positive slope; this corresponds to $\frac{d^3}{dz^3}\Pi_z > 0$

We have utilized the following formulas (power series up to 7-th power of z)

$$A(u) = G(u) - u$$

$$\Pi_z = \Pi_{z=0} + \sum_{n=1}^{\infty} a_n z^n$$

$$a_n = \frac{(-1)^{n-1}(2J)}{(n-1)!} \int_0^{u_1} \mathrm{d}u A(u) \left[1 + \frac{1}{n}\left(\ln P_0 + \int_0^u \mathrm{d}u' A(u')\right)\right] \left[\ln P_0 + \int_0^u \mathrm{d}u' A(u')\right]^n$$

$$P_0 = \frac{1}{D \int_0^{u_1} \mathrm{d}u \exp\left[\int_0^u \mathrm{d}u' A(u')\right]}$$

$$\int_0^{u_1} \mathrm{d}u A(u) = 1$$

If $a = 0.8, D = 0.1$ then the looked-for zeroes of $\frac{d^2}{dz^2}\Pi_z$ which cross the zero line with vertical slope are found:

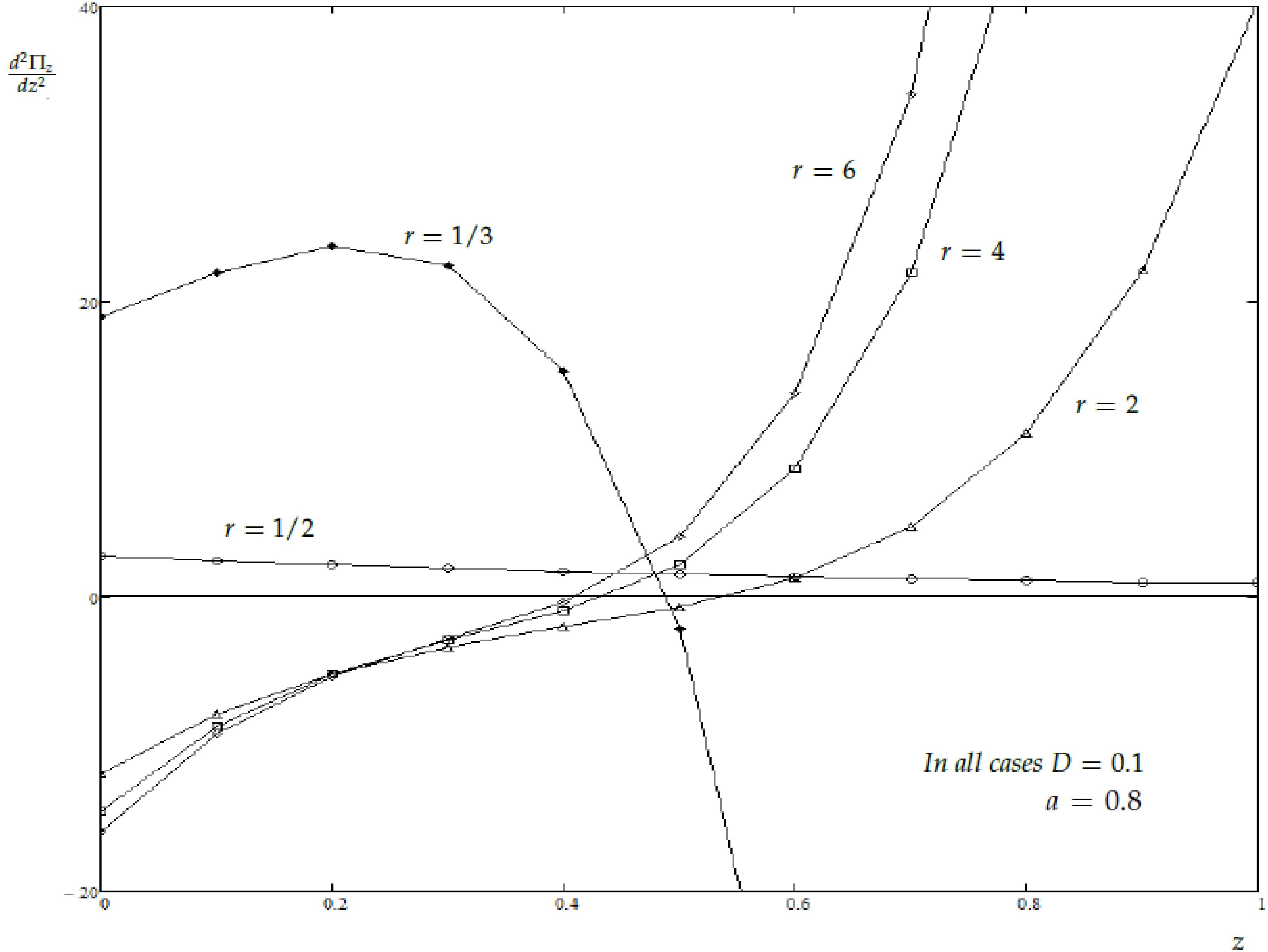
for $r = 2$ (at $z = 0.452$), corresponding to a power law with exponent $\frac{1}{0.452} = 2.21$

for $r = 4$ (at $z = 0.438$) , corresponding to a power law with exponent $\frac{1}{0.438} = 2.28$

for $r = 6$ (at $z = 0.412$) corresponding to a power law with exponent $\frac{1}{0.412} = 2.43$

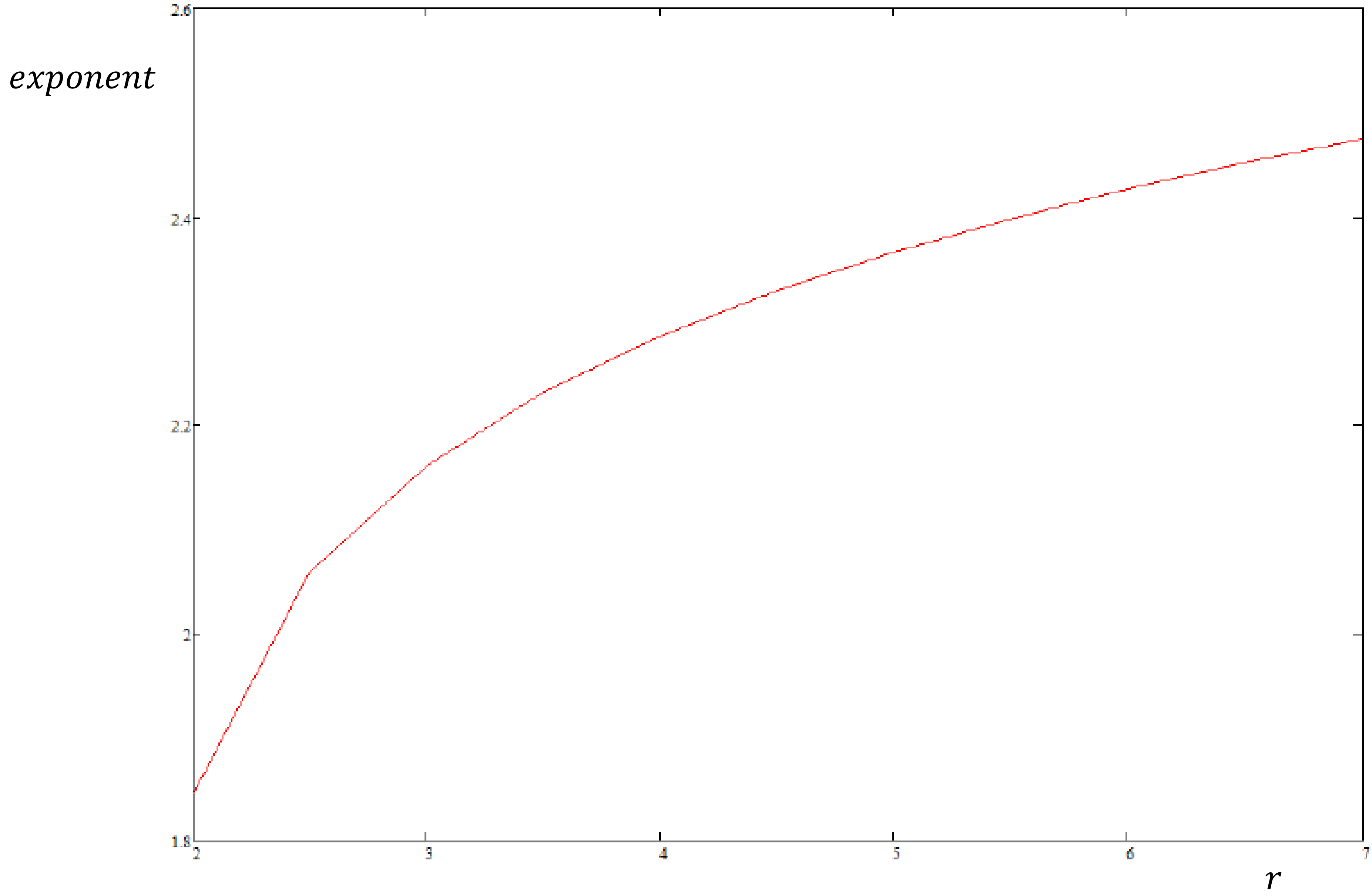
No such zeroes are found for values $r < 1$ of r, which correspond therefore to exponentials. (This makes sense, as Brownian motion is retrieved for $r \to 0$).

$\to$ Variance of fluctuations is larger for $r > 1$ than for $r < 1$

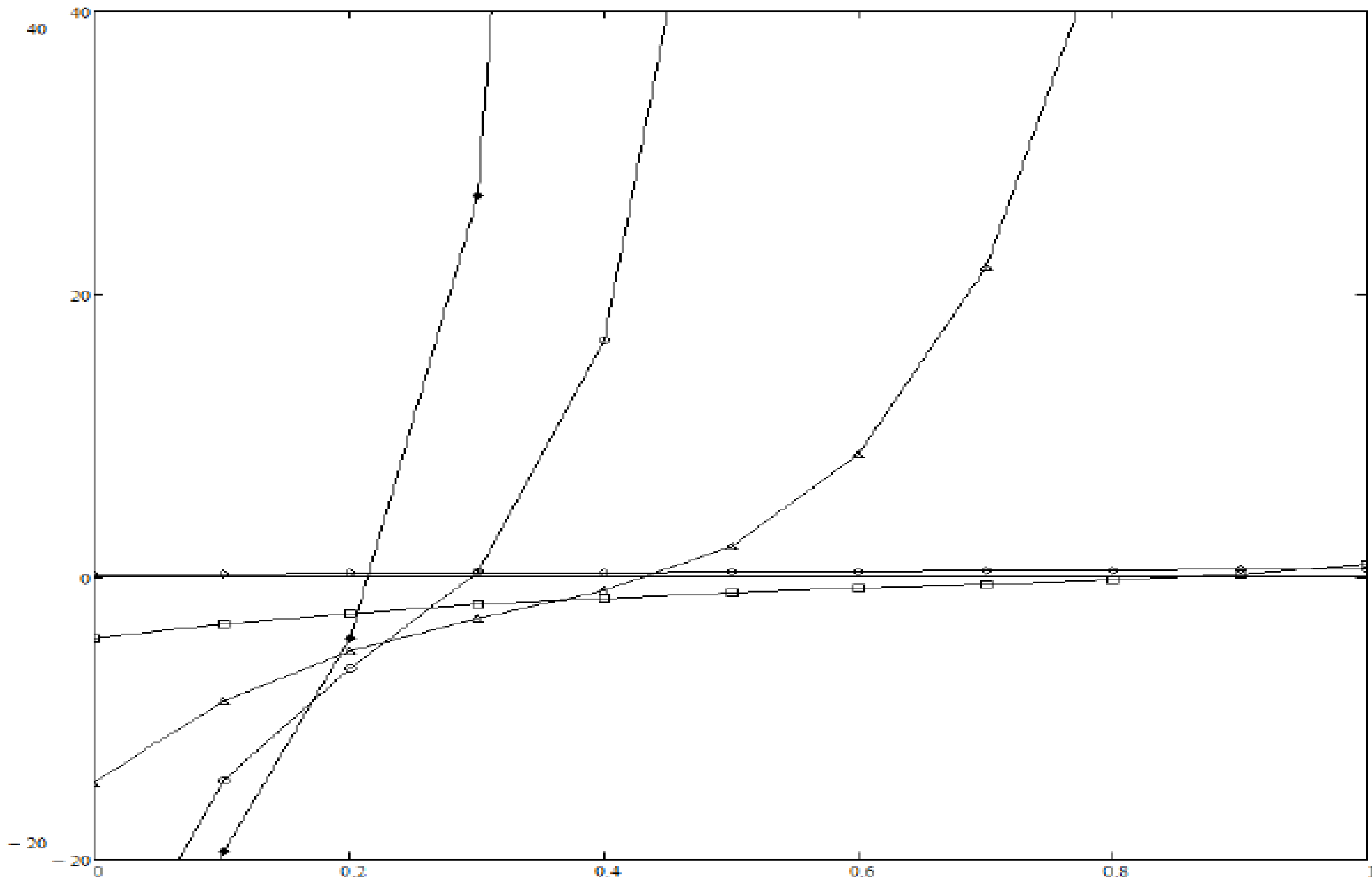


$\frac{d^2\Pi_z}{dz^2}$
40
20
0
– 20
0
0.2
0.4
0.6
0.8
1
z
$r = 6$
$r = 4$
$r = 2$
$r = 1/3$
$r = 1/2$
In all cases $D = 0.1$
$a = 0.8$

As r grows, the exponent of the power law saturates



The larger the noise, the larger D, the easier the relaxation to Boltzmann's distribution



$\frac{d^2\Pi_z}{dz^2}$ *(vertical axis) vs. z (horizontal axis) for D = 0.001 (black diamonds), D = 0.01 (empty circles), D = 0.1 (triangles), D = 2 (squares), D = 10 (empty diamonds). In all cases r = 4. Even if a relaxed state exists, the larger D, the stronger the noise, the nearer* z_c *to the bounds of the interval* $[0, 1)$. *If* $D > 1$ *then* z_c *does not belong to the interval, and Boltzmann's exponential distribution rules the relaxed state.*

Comparison with the results of Sànchez et al. 2007

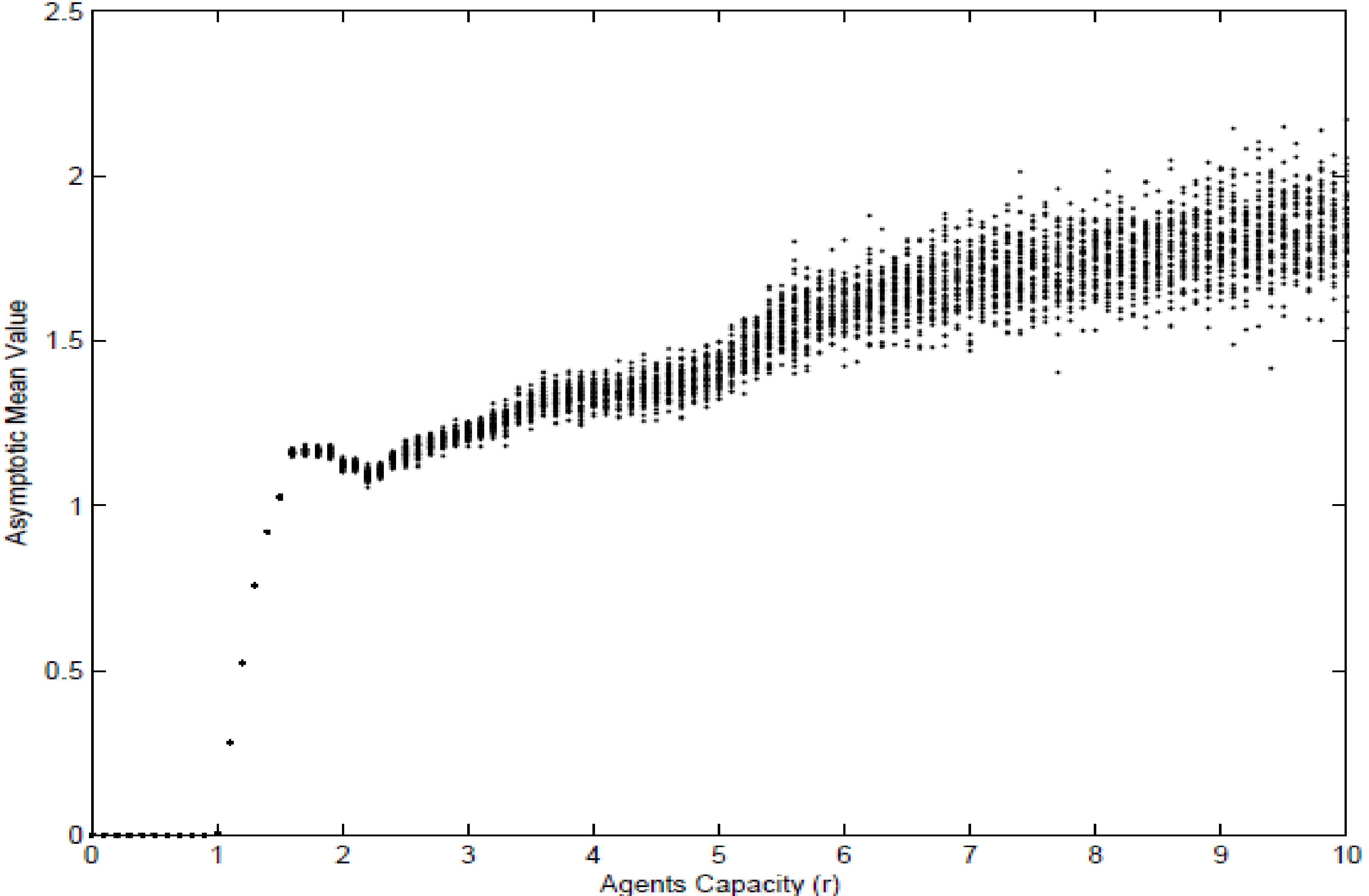
If $r > 1$: power law for with exponent 2.21 Pareto-like!

If $r < 1$: random fluctuations (around the $x = 0$ attractor of G)

Typical amplitude of fluctuations is much larger for $r > 1$ than for $r < 1$.

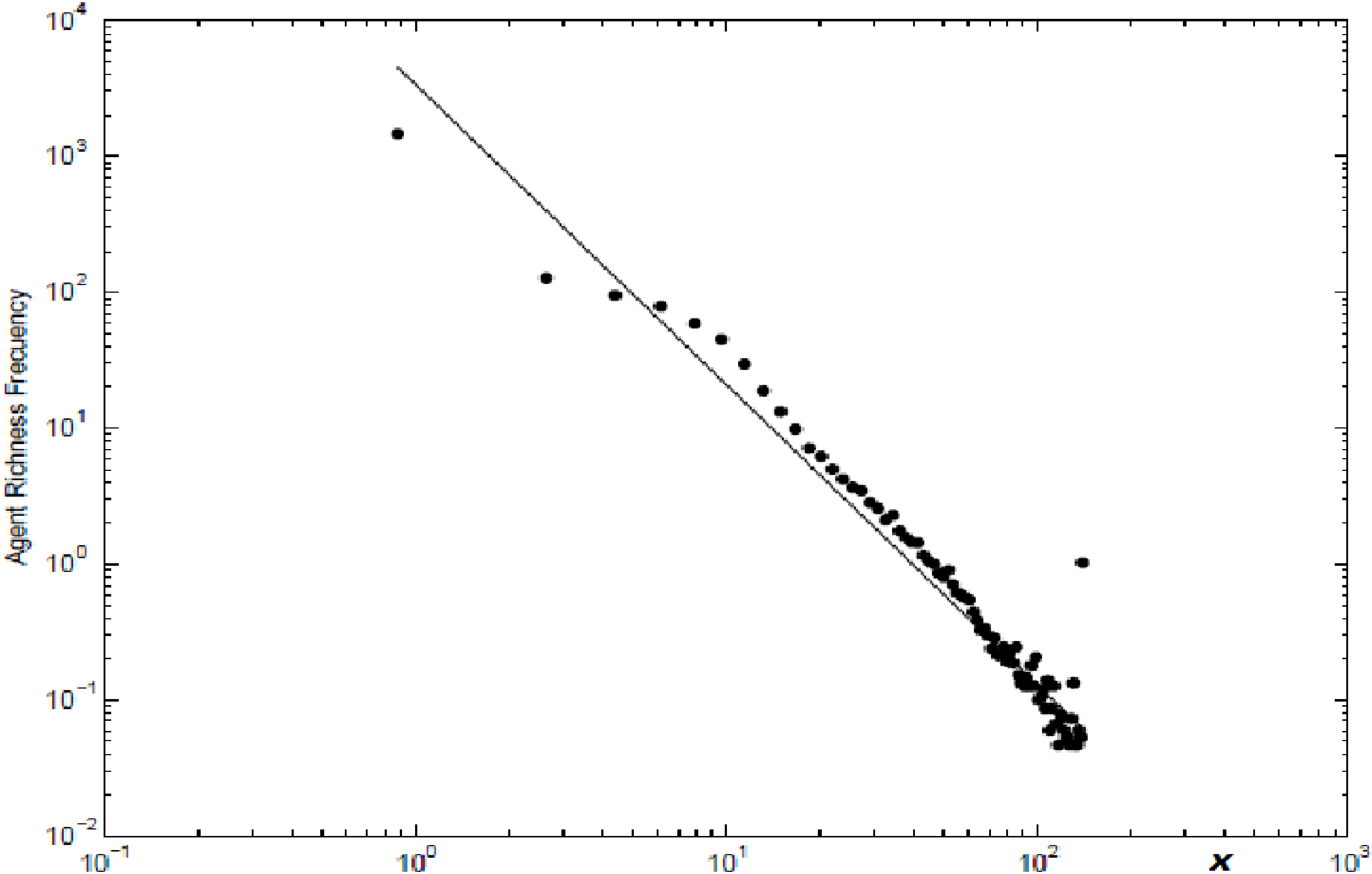
From Sànchez et al. 2007

For $i \to \infty$, fluctuations around mean value are much larger when $r > 1$



From Sànchez et al. 2007

$$P(x) \text{ for } i \to \infty \,;\, -\frac{d \ln P}{dx} = 2.21$$



Conclusions

Conclusions - I

Gibbs' thermodynamics describes the probability distribution of microstates in relaxed states, their stability against fluctuations and the process of relaxation with the help of Boltzmann's exponentials, Einstein's formula and Glansdorff et al.'s general evolution criterion (GEC) respectively.

Tsallis' thermodynamics describes the probability distribution of microstates in relaxed states with the help of q-exponentials ($\rightarrow$ power laws). Non-additivity prevents it from going further. Moreover, q is unknown, and is usually postulated - or obtained via lengthy numerical solution of the equations of motion.

Mapping of Tsallis' entropy onto an additive quantity with the same concavity allows generalization of both Einstein- and GEC-based conclusions to $q \neq 1$

Thus, relaxed states (if any exist) have to enjoy the same properties regardless of q – and the same is true for the relaxation processes leading to such states.

Conclusions - II

If a relaxed state exists, then $q \neq 1$ Einstein's rule and GEC allow us to identify the most stable probability distribution of microstates in a relaxed state (i.e., the probability distribution which the fluctuations of the largest amplitude relax to) as the q-exponential whose $q = 1 - z \in (0,1)$ minimizes $\frac{d}{dz}\Pi_z$, where Π_z is the amount of Tsallis' entropy produced per unit time in the bulk of the relaxed system.

If no such q exists, then the most stable probability distribution of microstates in the relaxed state (if any exists) is a Boltzmann exponential.

Explicit expressions for Π_z and its derivatives are provided for in the particular case of a system described by a 1D, nonlinear Fokker-Planck (NLFP) equation and weakly interacting with the external world. These expressions require just the knowledge of the diffusion coefficient and of the driving force acting in the NLFP.

Conclusions - III

We associate our NLFP with the stochastic equation obtained in the continuous limit from a 1D, autonomous map affected by noise. Relaxed solutions of NLFP (if any exists) $\leftrightarrow$ the asymptotic ($i \to \infty$) probability distributions $P(Q_i)$ (if any exists) for the outcome Q_i of the noise-affected map. Once the level of noise and the map dynamics are known, we may unambiguously obtain our NLFP and compute its diffusion coefficient and its driving force as well as Π_z and its derivatives.

Regardless of the nature (additive vs. multiplicative) of the noise, if $P(Q_i)$ exists then:

a) If z exists such that $0 < z < 1$ and $\frac{d}{dz}\Pi_z$ = min then $P(Q_i)$ is a q-exponential with $q = 1 - z$ (similar to a power law with exponent z^{-1});
b) Otherwise, $P(Q_i)$ is a Boltzmann's exponential.

In all cases, variance of fluctuations around a power-law distribution are always larger than around a Boltzmann's exponential.

Agreement with Pareto-like simulations of Sànchez et al. 2007 . No eqs. of motion solved!

From 1D to 2D maps and beyond… ?