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Generalized Entropies Depending Only on the Probability and Their Quantum Statistics

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Abstract: Modified entropies have been extensively considered in the literature [1]. Among the most well known are the Rényi entropy [2] and the Havdra-Charv$\widehat{'a}$ [3] and Tsallis entropy [4,5]. All these depend on one or several parameters. By means of a modification to Superstatistics [6], one of the authors [7] has proposed generalized entropies that depend only on the probability [7,8]. There are three entropies: $S_I = k\sum_{l=1}^{\Omega}(1-p_l^{p_l})$, $S_{II} = k\Sigma_{l=1}^{\Omega}(p_l^{-p_l}-1)$ and their linear combination $S_{III} = k\Sigma_{l=1}^{\Omega}\frac{p_l^{-p_l}-p_l^{p_l}}{2}$. It is interesting to notice that the expansion in series of these entropies having as a first term $S = -k\Sigma_{l=1}^{\Omega}p_l \ln p_l$ in the parameter $x_l \equiv p_l \ln p_l \leq 1$ cover, up to the first terms, any other expansion of any other possible function in x_l, one would want to propose as another entropy. The three proposed entropies in [7,8] are then the only possible generalizations of the Boltzmann-Gibbs (BG) or Shannon entropies that depend only of the probability. One obtains a superposition of two statistics (that of β and that of p_l), hence the name superstatistics. One may define an averaged Boltzmann factor as $B(E) = \int_0^{\infty} f(\beta)e^{\beta E}d\beta$ where $f(\beta)$ is the distribution of β. This work will deal with the analysis of the first two generalized entropies and will propose and deduce their associated quantum statistics; namely Bose-Einstein and Fermi-Dirac. The results will be compared with the standard ones and those due to the entropies in [3,4]. It will be seen in both cases that the BEO (the Bose-Einstein statistics corresponding to the entropies proposed by Obregón [7]) statistic differs slightly from the usual BE statistic and in the same way for FDO the difference is small from the usual FD.

Keywords: entropy; quantum statistic; Bose-Einstein; Fermi-Dirac; nonextensive entropy

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1. Introduction

In quantum statistics, Bose-Einstein statistics (BE statistics) is one of two possible ways in which a collection of non-interacting indistinguishable particles may occupy a set of available discrete energy states, at thermodynamic equilibrium. The theory was developed (1924–25) by Satyendra Nath Bose, the idea was later adopted and extended by Albert Einstein in collaboration with Bose.

The Bose-Einstein (BE) statistics apply only to those particles not limited to single occupancy of the same state that is, particles that do not obey the Pauli exclusion principle restrictions. Such particles have integer values of spin and are named bosons, after the statistics that correctly describe their behavior. There must also be no significant interaction between the particles.

On the other hand, in quantum statistics, a branch of physics, Fermi-Dirac statistics describe a distribution of particles over energy states in systems consisting of many identical particles that obey the Pauli exclusion principle. It is named after Enrico Fermi and Paul Dirac, each of whom discovered the method independently.

Fermi-Dirac (FD) statistics apply to identical particles with half-integer spin in a system with thermodynamic equilibrium. Additionally, the particles in this system are assumed to have negligible mutual interaction, that allows the many-particle system to be described in terms of single-particle energy states. The result is the FD distribution of particles over these states which includes the condition that no two particles can occupy the same state; this has a considerable effect on the properties of the system. Since FD statistics apply to particles with half-integer spin, these particles are therefore called fermions. It is most commonly applied to electrons, which are fermions with spin 1/2. Fermi-Dirac statistics are a part of the more general field of statistical mechanics and use the principles of quantum mechanics. The Bose-Einstein (BE), Fermi-Dirac (FD) and Classical (Boltzmann) distributions are given by $n_{BE} = \frac{1}{e^{(E_j-\mu)/kT}-1}$, $n_{FD} = \frac{1}{e^{(E_j-\mu)/kT}+1}$ and $n_{Cl} = e^{-(E_j-\mu)/kT}$ respectively, see Figure 1.

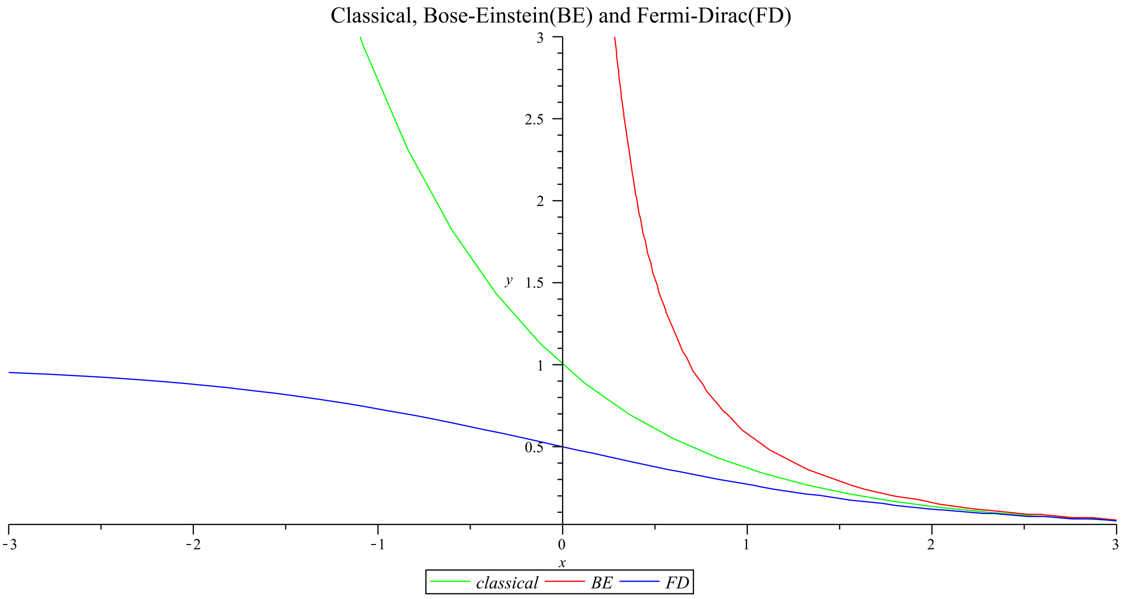

Figure 1. Comparison of Classic, BE and FD distributions.

The nonextensive entropy studied and developed by Tsallis [4], was previously defined by Havrda-Charvát [3]. Tsallis himself concedes that "Tsallis entropy" is a new rediscovery (not a discovery) in the labyrinthic history of entropies [5]. In recent years the nonextensive statistical mechanics, based on Tsallis entropy and the corresponding deformed exponential function, has been developed and attracted a lot of attention with a large amount of applications in rather diversified fields. Tsallis non-extensive statistical mechanics is a generalization of the Boltzmann-Gibbs (BG) statistical mechanics, this depends on a parameter q. In Section 2 we will review Tsallis entropy, analyzing the dependence of the parameter q on the mentioned entropy. Furthermore, as far as the nonextensive quantum statistical mechanics is concerned, in some places, the generalization to the BE distribution for bosons and FD for fermions have been investigated.

In [7,8] it is proposed a nonextensive statistical mechanics entropy that depends only on the probability distribution and not on a parameter in the framework of superstatistics; it is based on a Γ or χ^2 distribution that depends on β and also on p_l. The probabilities were calculated from the Boltzmann factor and show that it is possible to obtain the generalized entropy $S_I = k\sum_{l=1}^{\Omega} s(p_l)$, where $s(p_l) = 1 - p^{p_l}$; by maximizing this information measure, $p_l \equiv g_I(\beta E_l)$ for S_I and $p_l \equiv g_{II}(\beta E_l)$ for S_{II} are calculated as an implicit functions of βE_l and, at this stage of the procedure, p_l can be identified with the probability distribution, which we will review in Section 3.

It is interesting to study the extent of these two statistics with generalized entropies, in particular Havdra-Charvát (Tsallis). A direct generalization is the proposal $n(x) = \frac{1}{e^{x}_{2-q}\pm 1}$ [10]. We will assume a similar "corresponding" expression $n = \frac{1}{g^{-1}(\beta E_l)\pm 1}$. In Section 4 we will study this proposal of the generalization of the BE and FD statistics based on the two entropies S_I and S_{II}.

Given that $g_{_{I,II}}(\beta E_l)$ are only implicitly known, we will invert $\beta E_l(g_{_{I,II}})$ and, generalize for the number of occupation $n_{_{I,II}}$, taking the values of this function. We will calculate the generalized statistics corresponding to BE and FD namely $n_{_{I,II}} = \frac{1}{g^{-1}_{I,II}(\beta E_l) \pm 1}$. A brief summary is then presented in Section 5.

2. Results

2.1. Reviewing the q-Deformed Bose-Einstein (BE) and Fermi-Dirac (FD)

Tsallis nonextensive statistical mechanics is a generalization of the common Boltzmann-Gibbs (BG) statistical mechanics by postulating a generalized entropy of the classical one, $S = -k\sum_{l=1}^{\Omega} p_l \ln p_l$ to $S_q = -k\sum_{l=1}^{\Omega} p_l^q \ln_q p_l$, where k denotes Boltzmann constant. For simplicity we take $k = 1$, so the q-logarithm is $\ln_q x = \frac{x^{1-q}-1}{1-q}$ and its corresponding q-exponential function $e_q^x = (1+(1-q)x)^{1/(1-q)}$, where Ω gives the total number of the microscopic configuration in the system, and $q \in \mathbb{R}$ is the so-called nonextensive parameter. One can check that one recovers the BG statistics when $q \to 1$.

Furthermore, as far as the nonextensive quantum statistical mechanics is concerned, the generalized Bose Einstein distribution for bosons and Fermi Dirac distribution for fermions have been already studied, and it has been shown that a possible distribution function in nonextensive quantum statistics can be written as $\bar{n}_{_l} = 1/(e_{2-q}^{\alpha+\beta E_l} \pm 1)$, see [10]. In Figures 2 and 3 we show for different values of q the Tsallis Bose-Einstein and Fermi-Dirac distributions respectively.

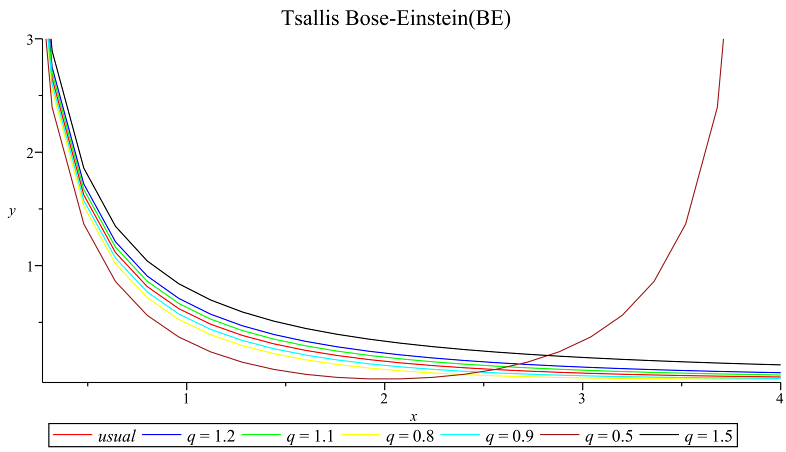

Figure 2. Comparison of Tsallis distributions (BE-Tsallis) for different values of q.

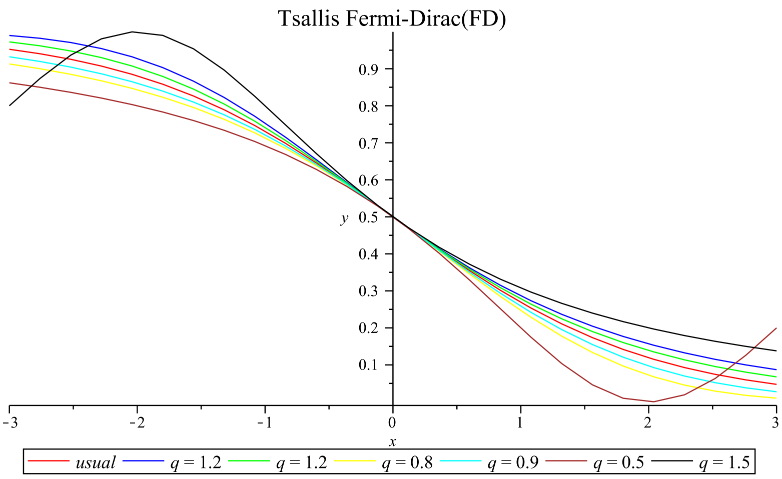

Figure 3. Comparison of Tsallis distributions (FD-Tsallis) for different values of q.

2.2. The Entropies Depending Only on the Probability Distribution

As already mentioned in [7,8] nonextensive statistical mechanics entropies have been proposed, that depend only on the probability distribution, and not on a parameter, in the framework of superstatistics. It is based on a Γ or (χ^2) distribution that depends on β and also on p_l. Then the Boltzmann factor is calculated and it is shown that it is possible to obtain the generalized entropies $S_I = k\sum_{l=1}^{\Omega} s_I(p_l)$, where $s_I(p_l) = 1 - p_l^{p_l}$ and $S_{II} = k\sum_{l=1}^{\Omega} s_{II}(p_l)$, where $s_{II}(p_l) = p_l^{-p_l} - 1$. By maximizing the corresponding information measure, p_l is implicitly expressed as function of βE_l and, at this stage of the procedure, p_l can be identified with the probability distribution.

In [8] it was shown that from the functional

$$\Phi = \frac{S}{k} - \gamma \sum_{l=1}^{\Omega} p_l - \beta \sum_{l=1}^{\Omega} p_l^{p_l+1} E_l,$$

maximizing Φ, p_l is obtained for S_I as

$$1 + \ln p_l + \beta E_l (1 + p_l + p_l \ln p_l) = p_l^{-p_l}. \quad (1)$$

And in a similar way for S_{II} as

$$1 + \ln p_l + \beta E_l (1 - p_l - p_l \ln p_l) = p_l^{p_l}. \quad (2)$$

The dominant term in these expressions correspond to the Boltzmann-Gibbs prediction, $p_l = e^{-\beta_o E_l}$. In general, however, we cannot analytically express p_l as function of βE_l.

In Figure 4, p_l is drawn as a function of the reduced energy βE_l. We notice that for relative large values of βE_l the usual values for p_l coincide with the ones given by Equations (1) and (2). As expected, they coincide also for p_l~1.

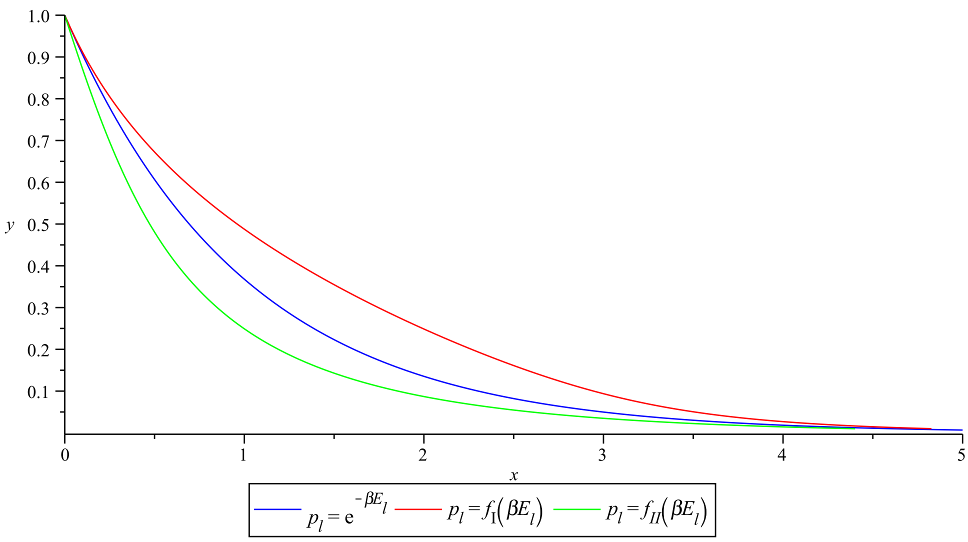

Figure 4. Comparison of the three probabilities. The blue line corresponds to the standard one $p_l = e^{-\beta E_l}$, red line to $p_l = g_I(\beta E_l)$ Equation (1), and green line $p_l = g_{II}(\beta E_l)$, Equation (2).

As mentioned direct generalization of the Havdra-Charvát or Tsallis entropy is given by $n(x) = \frac{1}{e^x_{2-q} \pm 1}$ [11,12]. The generalizations proposed here correspond to $n_{I,II} = \frac{1}{g^{-1}_{I,II}(\beta E_l) \pm 1}$. We will discuss them in the next section.

3. BEO and FDO, Generalized Distribution Functions

In this section we will discuss the extension to the nonextensive statistical mechanics whose entropies depend only on the probability [7,8]; corresponding to Bose-Einstein (BEO) and Fermi-Dirac (FDO) distributions. These entropies are: $S_I = k\sum_{l=1}^{\Omega}(1-p_l^{p_l})$, $S_{II} = k\sum_{l=1}^{\Omega}(p_l^{-p_l}-1)$. As stated to them correspond $n_{I,II} = \frac{1}{g_{I,II}^{-1}(\beta E_l)\pm 1}$.

We will take $\beta E_l(p_l)$ and invert it and get the values of $p_{l_{I,II}} \equiv g_{I,II}(\beta E_l)$. In this manner we will be able to calculate the occupancy number $n_{I,II}$ corresponding to the two entropies and their associated BEO and FDO.

It is shown that in both cases the BEO statistics differs from the usual BE statistics only slightly Figures 5 and 6.

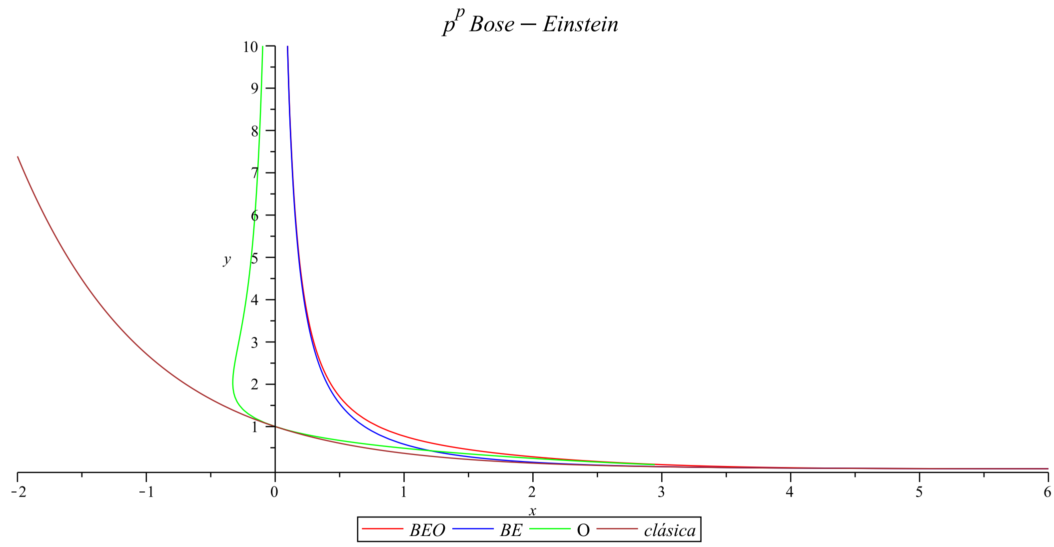

Figure 5. Comparison of BE with BEO (n_I); and classical $p_l = e^{-\beta E_l}$ with $g_I(\beta E_l)$. The red line corresponds to BEO, blue line to BE usual, green line to $g_I(\beta E_l)$ and brown line to classical (Boltzmann).

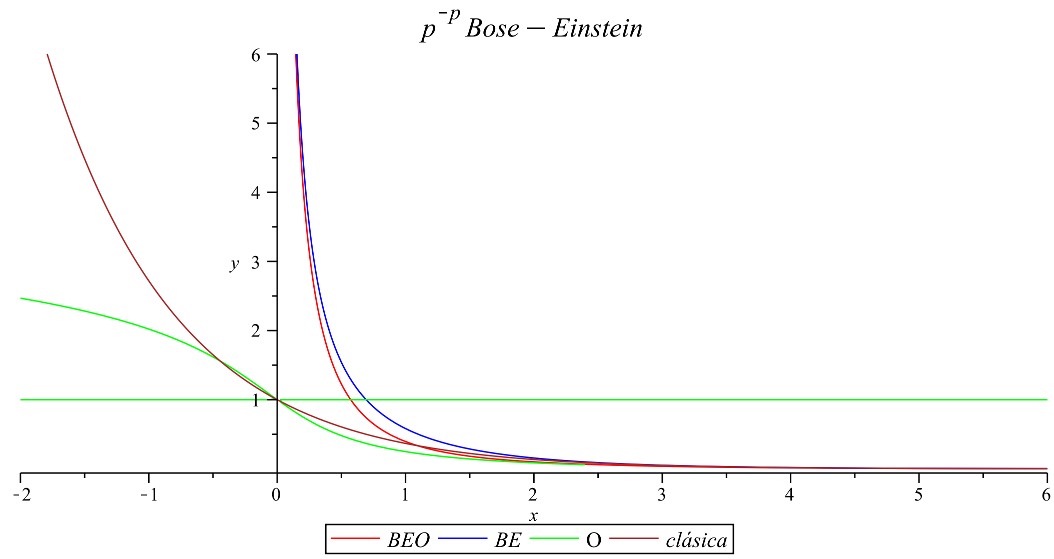

Figure 6. Comparison of BE with BEO (n_{II}), and classical $p_l = e^{-\beta E_l}$ with $g_{II}(\beta E_l)$. The red line corresponds to BEO, blue line to BE usual, green line to $g_{II}(\beta E_l)$, brown line to classical (Boltzmann).

In the same way we can see that the FDO statistics differs from the usual FD statistics in a small amount, Figures 7 and 8.

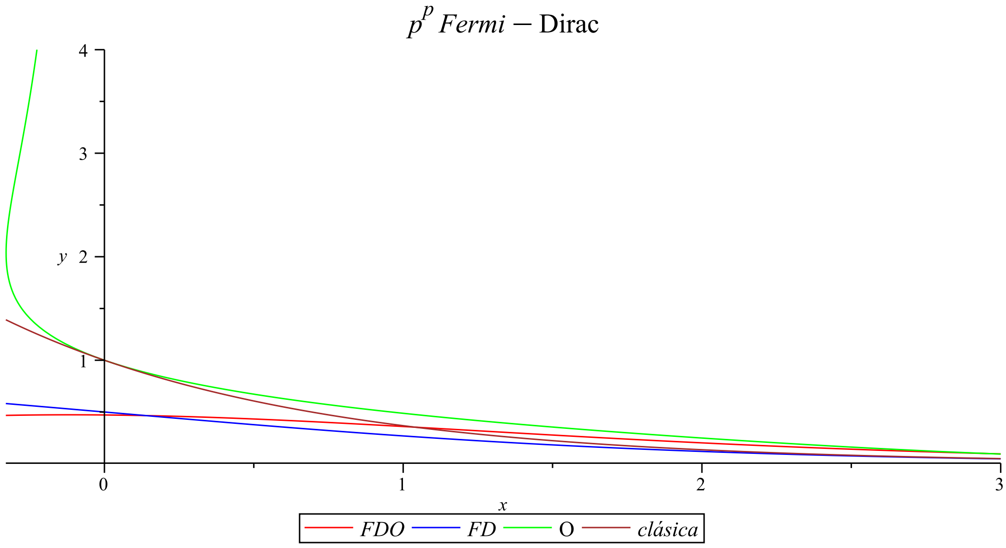

Figure 7. Comparison of FD with FDO (n_I), and classical $p_l = e^{-\beta E_l}$ with $g_I(\beta E_l)$. The red line corresponds to FDO, blue line to FD usual, green line to $g_I(\beta E_l)$, brown line to classical (Boltzmann).

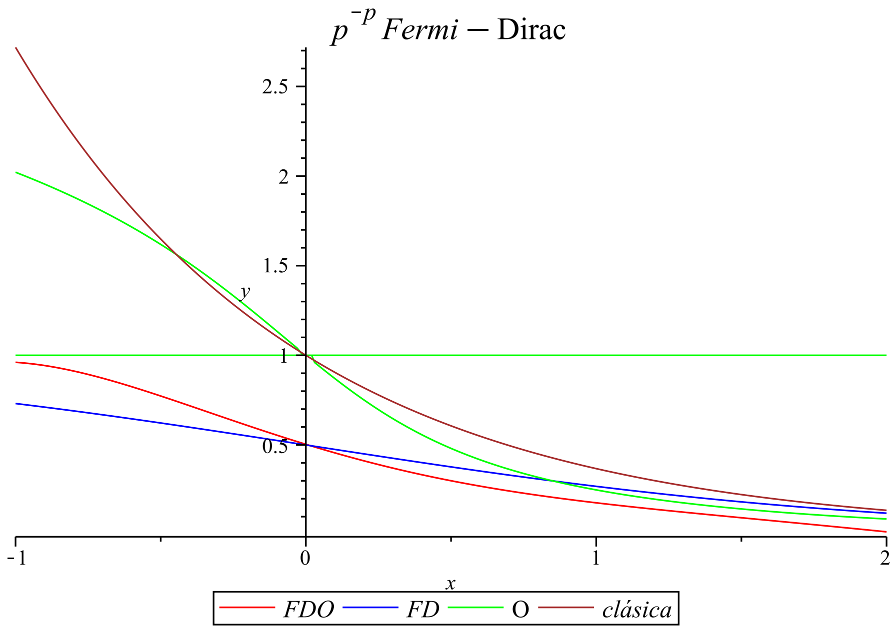

Figure 8. Comparison of FD with FDO (n_{II}), and classical $p_l = e^{-\beta E_l}$ with $g_{II}(\beta E_l)$. The red line corresponds to FDO, blue line to FD usual, green line to $g_{II}(\beta E_l)$, brown line to classical (Boltzmann).

We observe that the usual BE and FD occupancy numbers are only slightly modified when considering the generalized entropies S_I and S_{II}. This happens also in the q-entropy [3,4] for q near to one.

4. Conclusions

We started by making a quick revision of the nonextensive statistical mechanics of Tsallis [4] in which one recovers the usual statistical mechanics for $q = 1$ (Boltzmann). Then we reviewed the proposed occupancy number for the BE and FD q-statistics comparing them with the usual ones. For the nonextensive statistical mechanics which entropies depend only on the probability, we make a proposal also based on what it should be a generalized exponential, namely the inverse of the probability. This can only be performed numerically, if $p_{l_{I,II}} \equiv g_{I,II}(\beta E_l)$ the proposal is $n_{I,II} = \frac{1}{g^{-1}_{I,II}(\beta E_l)\pm 1}$, we showed that the BEO and FDO occupancy numbers differ only slightly from the usual ones. This happens also for the q-entropies, for q near to one.

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