UNIVERSITÄT ZU LÜBECK
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Theory and Practice of Permutation Entropy

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Contents

1. Motivation

- Entropies and entropy-like quantities are playing an increasing role in data analysis, in the contexts of dynamical systems and of stochastic processes.
- Various applications of entropies are given, for example, in the analysis of physiological time series.
- Central classical concepts: Approximate entropy, Sample entropy, and variants.
- Interesting relatively new concepts: **Permutation entropies** and variants based on the ordinal structure of time series and systems behind them.
- Important: **good theoretical understanding** of those concepts and their relationship to other entropies, and their **adequate application** to the analysis of data.

2. Context

Modeling (one observable, simplest setting)

time series $(x_t)_{t=0}^{N-1} = (x_0, x_1, x_2, \ldots, x_{N-1})$ assumed as values measured on the orbit of a dynamical system

- $(\Omega, \mathcal{A}, \mu, T)$ μ-**preserving dynamical system**
- X real-valued random variable on Ω
- **observable** X provides 'outreading' stochastic process $(X \circ T^{\circ t})_{t=0}^{\infty}$ with realizations $(X \circ T^{\circ t}(\omega))_{t=0}^{\infty}$; $\omega \in \Omega$

assumptions:

- **Ergodicity:** realizations represent distribution of X
- no information loss by the measuring process with respect to X ('separation' of orbits), natural[1]

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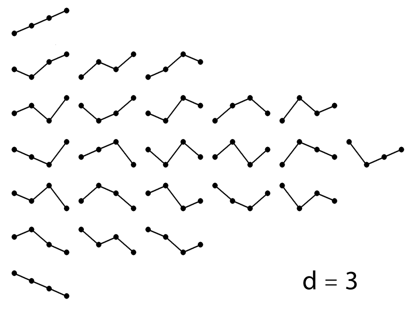
[1] Gutman, Takens embedding theorem with a continuous observable, in: Proceedings of the Erg. Th. Workshops UNC Chapel Hill 2013-2014. De Gruyter

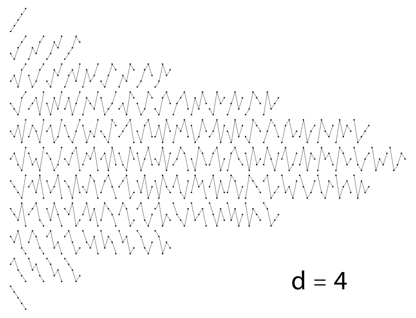
Ordinal patterns

Definition

$(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}$ has the **(ordinal)** d-**pattern** $\pi = (r_0, r_1, \ldots, r_d) \in \Pi_d$ if

$$x_{r_0} \geq x_{r_1} \geq \ldots \geq x_{r_d}, \text{ and}$$
$$r_{l-1} > r_l \text{ in the case } x_{r_{l-1}} = x_{r_l}.$$





Relative frequencies / probabilities of ordinal d-pattern words

Definition

- **relative frequency of d-patt. word** $(\pi_1, \pi_2, \ldots, \pi_k) \in \Pi_d^k$ in time series $(x_t)_{t=0}^{N-1}$:

$$p_{(\pi_1,\pi_2,\ldots,\pi_k)} := \frac{1}{N-d-k+1} \#\{s \in \{0,\ldots,N-d-k\} \mid \text{for all } i = 1,\ldots,k \; (x_{s+i-1}, x_{s+i} \ldots, x_{s+i-1+d}) \text{ has ordinal pattern } \pi_i\}$$

- **probability of d-pattern word** $(\pi_1, \pi_2, \ldots, \pi_k) \in \Pi_d^k$ for $(\Omega, \mathcal{A}, \mu, T, X)$:

$$P_{(\pi_1,\pi_2,\ldots,\pi_k)} := \mu(\{\omega \in \Omega \mid \text{for all } i = 1,\ldots,k \; X \circ T^{\circ i-1}(\omega), X \circ T^{\circ i}(\omega), \ldots, X \circ T^{\circ d}(\omega) \text{ has ordinal pattern } \pi_i\})$$

- $p_{(\pi_1,\pi_2,\ldots,\pi_k)}$ estimates $P_{(\pi_1,\pi_2,\ldots,\pi_k)}$

3. From KS entropy to permutation entropy

Entropies (I)

Kolmogorov-Sinai entropy (**KS entropy**) is central theoretical complexity measure for dynamical systems, but not easy to determinate and to estimate.

ordinal approach:

- **entropy of** d**-pattern-**k**-words**:

$$H(d,k) := -\sum_{(\pi_1,\ldots,\pi_k)\in\Pi_d^k} P_{(\pi_1,\ldots,\pi_k)} \ln P_{(\pi_1,\ldots,\pi_k)}$$

- **empirical entropy of** d**-pattern-**k**-words**:

$$h(d,k) := -\sum_{(\pi_1,\ldots,\pi_k)\in\Pi_d^k} p_{(\pi_1,\ldots,\pi_k)} \ln p_{(\pi_1,\ldots,\pi_k)}$$

- $h(d,k)$ estimates $H(d,k)$
- $H(d,0) := 0$

Entropies (II)

- **entropy rate of** d**-patterns**:

$$\mathrm{EntroRate}(d) := \lim_{k\to\infty}(\downarrow)\frac{H(d,k)}{k} = \lim_{k\to\infty}(\downarrow)(H(d,k+1)-H(d,k))$$

Theorem

$$\text{KS entropy} = \lim(\uparrow)_{d\to\infty}\mathrm{EntroRate}(d) \leq \text{Permutation entropy}^{2}$$

[2] Antoniouk, K., Maksymenko, Discrete Contin. Dyn. Syst. A 34 (2014); K., Maksymenko,Stolz, Discrete Contin. Dyn. Syst. B 20 (2015);

- only one limit for **Permutation entropy**[3]:

$$\text{Permutation entropy} = \limsup_{d\to\infty}\frac{H(d,1)}{d} \qquad \left(\text{better: } \liminf_{d\to\infty}\frac{H(d,1)}{d}\right)$$

[3] Bandt, Pompe, Phys. Rev. Lett. 88 (2002), X identity

Entropies (III)

Remark

KS entropy = Permutation entropy for a special class of dynamical systems (piecewise monotone interval maps) and X identity[4]

[4] Bandt, G. Keller, Pompe, Nonlinearity 15 (2002)

General ordinal view on KS entropy

relevant entropies:

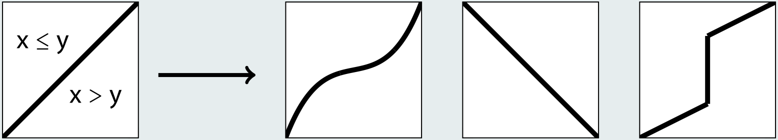
$$\begin{array}{cccccccc}
H(1,1) & H(1,2) & H(1,3) & H(1,4) & H(1,4) & H(1,5) & H(1,6) & \cdots \\
H(2,1) & H(2,2) & H(2,3) & H(2,4) & H(2,4) & H(2,5) & H(2,6) & \cdots \\
H(3,1) & H(3,2) & H(3,3) & H(3,4) & H(3,4) & H(3,5) & H(3,6) & \cdots \\
H(4,1) & H(4,2) & H(4,3) & H(4,4) & H(4,4) & H(4,5) & H(4,6) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

right weighting:

$$\begin{array}{cccccccc}
H(1,1) & H(1,2)/2 & H(1,3)/3 & H(1,4)/4 & H(1,5)/5 & \cdots & \searrow & \mathrm{EntroRate}(1) \\
H(2,1) & H(2,2)/2 & H(2,3)/3 & H(2,4)/4 & H(2,5)/5 & \cdots & \searrow & \mathrm{EntroRate}(2) \\
H(3,1) & H(3,2)/2 & H(3,3)/3 & H(3,4)/4 & H(3,5)/5 & \cdots & \searrow & \mathrm{EntroRate}(3) \\
H(4,1) & H(4,2)/2 & H(4,3)/3 & H(4,4)/4 & H(4,5)/5 & \cdots & \searrow & \mathrm{EntroRate}(4) \\
 & & & & & & & \vdots \\
 & & & & & & & \nearrow \\
 & & & & & & & \text{KS entropy}
\end{array}$$

Generalizations

- generalization to finitely (infinitely) many observables $X_1, X_2, \ldots, X_n(, \ldots)$
- generalization to many non-ergodic cases (ergodic decomposition)
- generalization to new two-dimensionel basic symbolization schemes[5]



[5] Stolz, Keller 2017

4. Conditional viewpoint

‘Conditional’ ordinal view on KS entropy

$$\begin{array}{cccccl}
H(1,2)-H(1,1) & H(1,3)-H(1,2) & H(1,4)-H(1,3) & \cdots & \searrow & \text{EntroRate}(1)\\
H(2,2)-H(2,1) & H(2,3)-H(2,2) & H(2,4)-H(2,3) & \cdots & \searrow & \text{EntroRate}(2)\\
H(3,2)-H(3,1) & H(3,3)-H(3,2) & H(3,4)-H(3,3) & \cdots & \searrow & \text{EntroRate}(3)\\
H(4,2)-H(4,1) & H(4,3)-H(4,2) & H(4,4)-H(4,3) & \cdots & \searrow & \text{EntroRate}(4)\\
 & & & & & \vdots\\
 & & & & & \nearrow\\
 & & & & & \text{KS entropy}
\end{array}$$

- For all natural numbers k it holds

$$H(d,k)-H(d,k-1) \le \frac{1}{k}\sum_{i=1}^{k}\left(H(d,i)-H(d,i-1)\right) = \frac{H(d,k)}{k}$$

Lemma

For each sequence $(k_d)_{d=1}^{\infty}$ of natural numbers it holds

$$\text{KS entropy} \le \liminf_{d\to\infty}\left(H(d,k_d+1)-H(d,k_d)\right) \le \liminf_{d\to\infty}\frac{H(d,k_d)}{k_d}$$

Conditional entropy of ordinal patterns[6]

- special case: $k_d = 1$ for all $d \in \mathbb{N}$:

Definition

conditional entropy of ordinal patterns

$$\liminf_{d\to\infty} H(d,2) - H(d,1)$$

- interpretation: uncertainty of next type of ordinal pattern of high order given one such type
- also interesting:

$$\liminf_{d\to\infty} H(d,k+1) - H(d,k)$$

for $k = 2, 3, 4, \ldots$

[6] Unakafov, K., Physica D 269 (2014), 94-102.

Conditional entropy better than PE?

assuming existence of

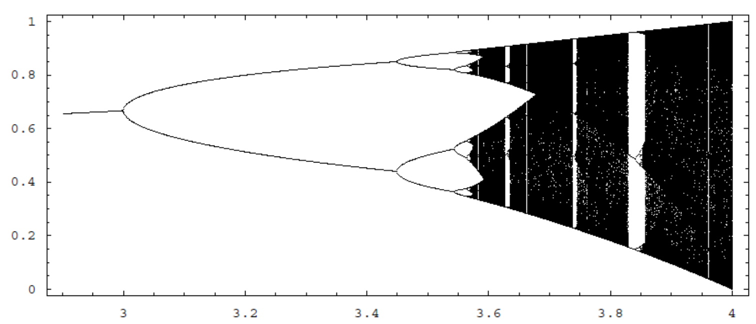
$$\lim_{d\to\infty}(H(d+1,1)-H(d,1)),$$

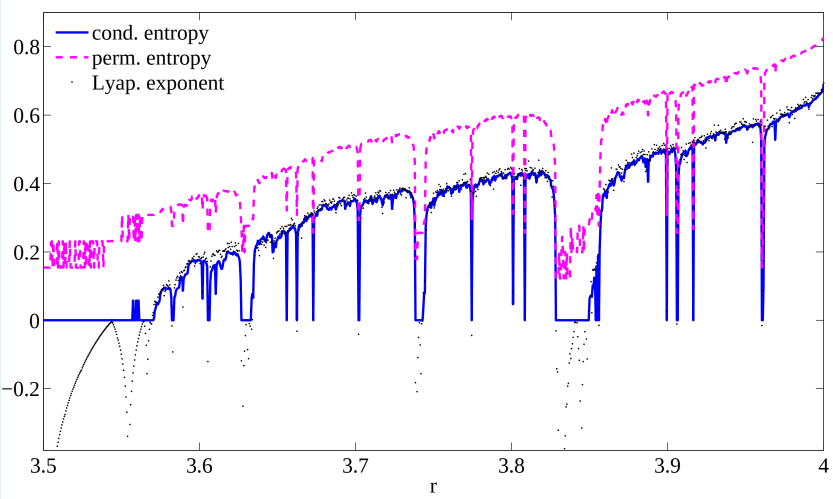
which looks natural, by the Stolz-Cesàro theorem with $H(d,0) := 0$ it holds

$$\begin{aligned}
\textbf{KS entropy} &\leq \liminf_{d\to\infty}(H(d,2)-H(d,1)) \\
&= \textbf{conditional entropy of ordinal patterns} \\
&\leq \lim_{d\to\infty}(H(d+1,1)-H(d,1)) \\
&= \lim_{d\to\infty}(H(d,1)-H(d-1,1)) \\
&= \lim_{d\to\infty}\frac{1}{d}\sum_{i=1}^{d}(H(i,1)-H(i-1,1)) \\
&= \liminf_{d\to\infty}\frac{1}{d}H(d,1) = \textbf{Permutation entropy}
\end{aligned}$$

Logistic family and permutation entropies

- $x \in [0,1] \mapsto rx(1-x)$ for different $r \in [0,4]$
- for almost all $r \in [0,4]$ the KS entropy either coincides with the Lyapunov exponent if it is positive or is equal to zero otherwise (Pesin's formula, 'natural' RB-measure)



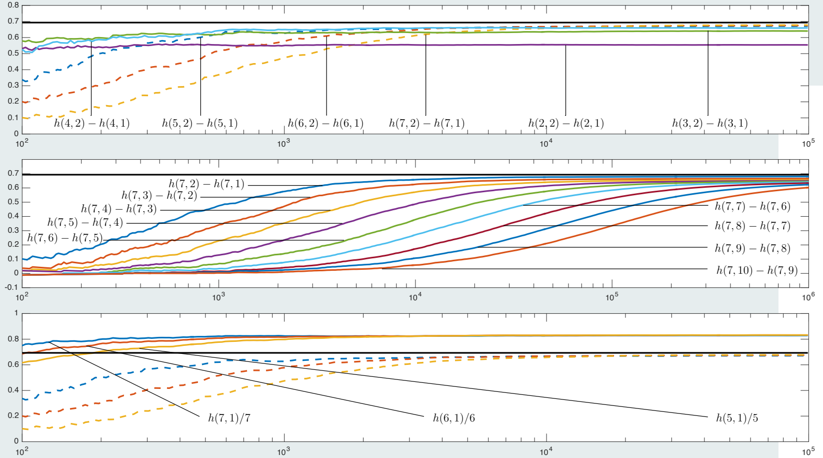


Lyapunov exponent, empirical and **conditional Permutation entropy** (d=9)

5. Practical aspects

Asymptotics versus statistics[7]

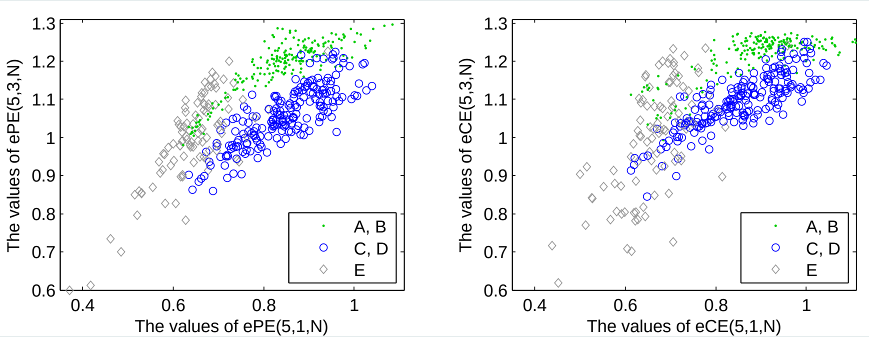
$\Omega = [0,1], T(\omega) = 4\omega(1-\omega), X = \text{id}$



7 Keller, Mangold, Stolz, Werner Entropy 19 (2017)

Combining entropies, delays

example: EEG recordings from the Bonn EEG Database[8]



empirical Permutation entropy versus empirical Permutation entropy/
conditional Permutation entropy versus conditional Permutation entropy,

also other entropies

8 available online: http://epileptologie-bonn.de

Variants of Permutation entropy

Reinventing metric information

- 'Flat' ordinal patterns, i.e. ordinal patterns coming from vectors of low variability, are very sensitive with respect to noise.

⇒ **robust Permutation entropy**[9] excluding 'flat' ordinal patterns

⇒ **weighted Permutation entropy**[10]: weighting ordinal patterns by variance of vectors behind

- theoretically not well understood

Generalization of conditional approach by Armand Eyebe Fouda, Koepf, Jacquir[11]

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[9] Keller, Unakafov, Unakafova, Entropy 16 (2014)

[10] Fadlallah, Chen, Keil, Prìncipe, Phys. Rev. E 87 (2013)

[11] Communications in Nonlinear Sci. and Num. Simulat. 46 (2017), 103-115

Further aspects

- generally: compromise between theoretical requirements and practical possibilities necessary
- requires better theoretical understanding of measures
- results in standardizations which have to be communicated and commonly used
- standards have to be (automatically) related to kind and size of data
- finding best measures also needs application of machine learning

Thank you!