Pointwise Partial Information Decomposition Using Specificity and Ambiguity Lattices

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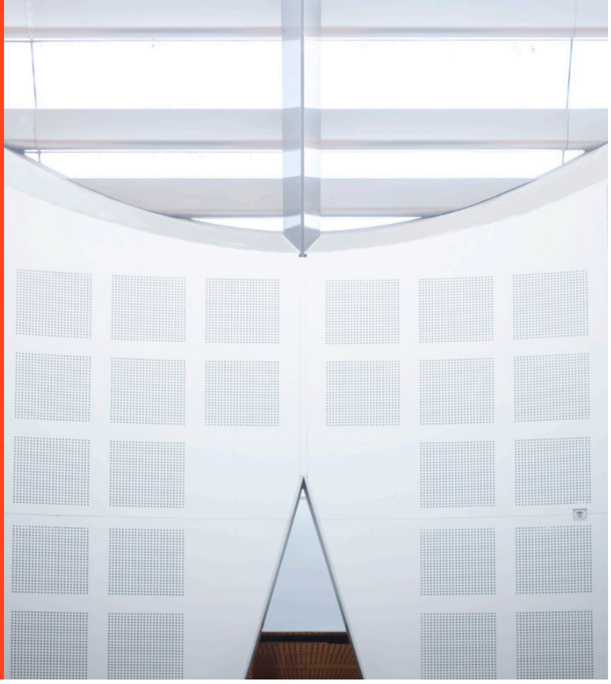


Information decomposition

Goal: identify redundant, unique and complementary (synergistic) components of MI

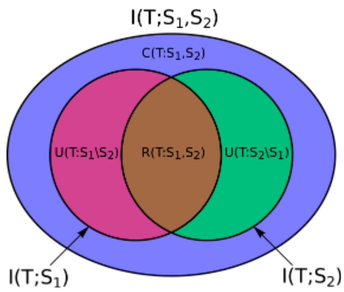
Consider three random variables S_1, S_2 and T

- Aim: predict T using S_1 and S_2
- Mutual information captures

$$I(T;S_1) = R(T:S_1;S_2) + U(T:S_1 \setminus S_2)$$
$$I(T;S_2) = R(T:S_1;S_2) + U(T:S_2 \setminus S_1)$$

- Joint mutual information captures

$$I(T;S_1;S_2) = R(T:S_1;S_2) + U(T:S_1 \setminus S_2) + U(T:S_2 \setminus S_1) + C(T:S_1;S_2)$$



Information decomposition: examples

1. Unique information $U(T:S_1 \setminus S_2)$
2. Redundant information $R(T:S_1;S_2)$
3. Synergistic information $C(T:S_1;S_2)$
4. In general, all types of information are present

UNQ

p	s_1	s_2	t
½	0	0	0
½	1	0	1

RDN

p	s_1	s_2	t
½	0	0	0
½	1	1	1

XOR

p	s_1	s_2	t
¼	0	0	0
¼	0	1	1
¼	1	0	1
¼	1	1	0

AND

p	s_1	s_2	t
¼	0	0	0
¼	0	1	0
¼	1	0	0
¼	1	1	1

Information decomposition

Goal: identify redundant, unique and complementary (synergistic) components of MI

We have 3 equations in 4 unknowns!

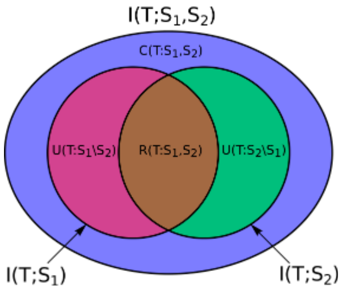
Consider three random variables S_1, S_2 and T

- Aim: predict T using S_1 and S_2
- Mutual information captures

$$I(T; S_1) = R(T : S_1; S_2) + U(T : S_1 n S_2)$$
$$I(T; S_2) = R(T : S_1; S_2) + U(T : S_2 n S_1)$$

- Joint mutual information captures

$$I(T; S_1; S_2) = R(T : S_1; S_2) + U(T : S_1 n S_2) + U(T : S_2 n S_1) + C(T : S_1; S_2)$$



Partial Information Decomposition (Williams and Beer, 2010)

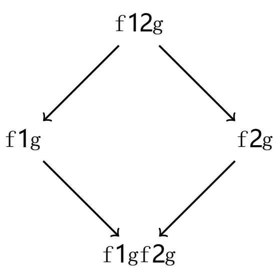
- Axiomatic framework extending this decomposition to arbitrary number of sources

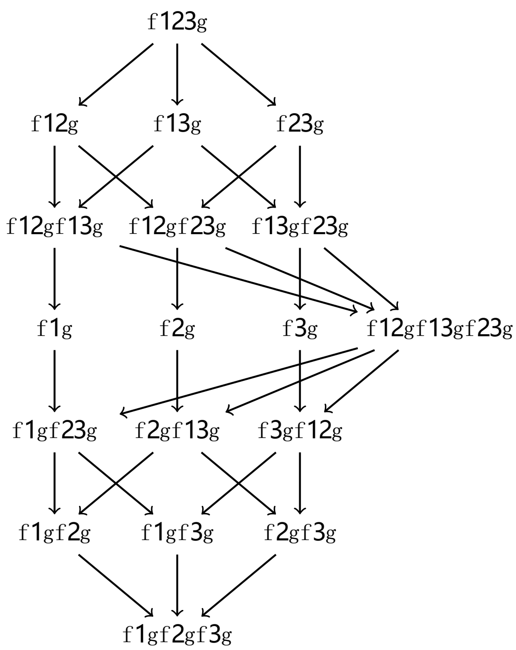
Axioms (PID)

(1) Symmetry: $R(T : S_1; :::; S_n)$ is invariant under permutations of the S_i's
(2) Monotonicity: $R(T : S_1; :::; S_n) \square R(T : S_1; :::; S_{n \square 1})$
(3) Self-redundancy: $R(T : S_i) = I(T; S_i)$

- Yields a redundancy lattice

Redundancy lattice





Partial Information Decomposition (Williams and Beer, 2010)

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(1) Symmetry: $R(T : S_1; \ldots; S_n)$ is invariant under permutations of the S_i's
(2) Monotonicity: $R(T : S_1; \ldots; S_n) \leq R(T : S_1; \ldots; S_{n-1})$
(3) Self-redundancy: $R(T : S_i) = I(T; S_i)$

- Yields a redundancy lattice
- Still no accepted, compatible definition of unique, redundant, or synergistic information

Pointwise information theory

- From four postulates, Fano (1961) derived the pointwise mutual information

$$i(x; y) = \log \frac{p(y|x)}{p(y)} = \log \frac{p(x|y)}{p(x)} \gtrless 0$$

- Corollaries: (average) mutual information, pointwise entropy and (Shannon) entropy

Pointwise information decomposition

- Pointwise decomposition for each realisation

$$\begin{aligned} i(t; s_1) &= r(t : s_1; s_2) + u(t : s_1 \setminus s_2) \\ i(t; s_2) &= r(t : s_1; s_2) + u(t : s_2 \setminus s_1) \\ i(t; s_1 s_2) &= r(t : s_1; s_2) + u(t : s_1 \setminus s_2) + u(t : s_2 \setminus s_1) + c(t : s_1; s_2) \end{aligned}$$

- Should be able to take the expectation over all realisations

$$\begin{aligned} R(T : S_1; S_2) &= \mathbb{E}\left[r(t : s_1; s_2)\right] \qquad & U(T : S_1 \setminus S_2) &= \mathbb{E}\left[u(t : s_1 \setminus s_2)\right] \\ C(T : S_1; S_2) &= \mathbb{E}\left[c(t : s_1; s_2)\right] \qquad & U(T : S_2 \setminus S_1) &= \mathbb{E}\left[u(t : s_2 \setminus s_1)\right] \end{aligned}$$

- This should recover the (average) information decomposition

$$\begin{aligned} I(T; S_1) &= R(T : S_1; S_2) + U(T : S_1 \setminus S_2) \\ I(T; S_2) &= R(T : S_1; S_2) + U(T : S_2 \setminus S_1) \\ I(T; S_1 S_2) &= R(T : S_1; S_2) + U(T : S_1 \setminus S_2) + U(T : S_2 \setminus S_1) + C(T : S_1; S_2) \end{aligned}$$

Motivation: PwUnq

- Consider PwUnq:

p	s_1	s_2	t	$i(t;s_1)$	$i(t;s_2)$	$i(t;s_1s_2)$	r	u_1	u_2	c
¼	0	1	1	0	1	1	0	0	1	0
¼	1	0	1	1	0	1	0	1	0	0
¼	0	2	2	0	1	1	0	0	1	0
¼	2	0	2	1	0	1	0	1	0	0
Expected values				½	½	1	0	½	½	0

- According to I_{min} Williams and Beer (2010), $\widetilde{UI}$ of Bertschinger et al. (2014), S_{VK} of Griffith and Koch (2014) and I_{red} of Harder et al. (2013)

$$R = \langle r \rangle = 1/2 \text{ bit} \neq 0 \text{ bit}$$

Pointwise Partial Information Decomposition

Axioms (PPID)

(1) Symmetry: $r(t : s_1; \ldots; s_n)$ is invariant under permutations of the s_i's

(2) Monotonicity: $r(t : s_1; \ldots; s_n) \leq r(t : s_1; \ldots; s_{n-1})$

(3) Self-redundancy: $r(t : s_i) = i(t; s_i)$

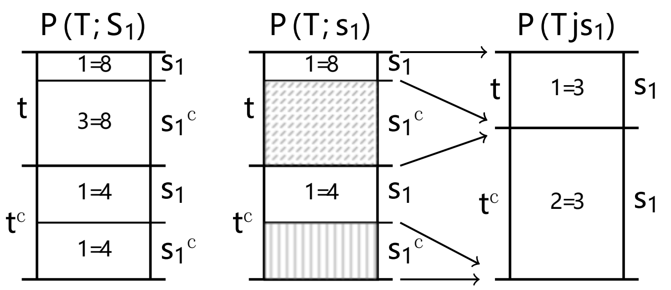
- Problems:
  1. Pointwise mutual information is not non-negative
  2. Still no clear definition of redundant information

What is pointwise information $i(t; s_1)$?

- The surprise of the posterior compared to the surprise of the prior for event t, s_1

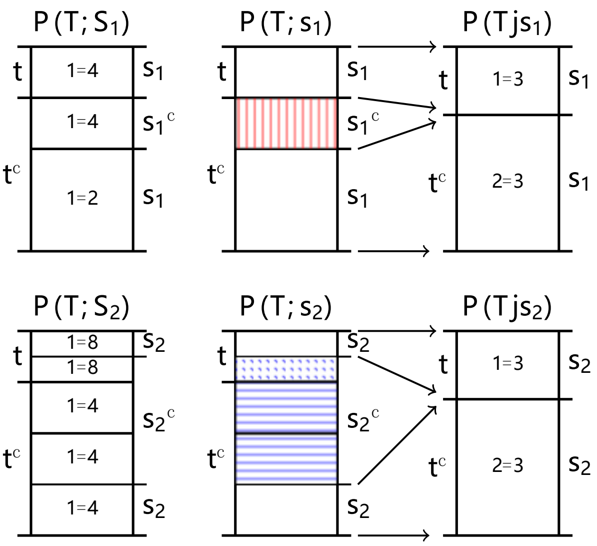
$$\text{Prior} \quad p(t) \longrightarrow p(t|s_1) \quad \text{Posterior}$$

- Finn et al. (2017)—this change is ultimately derived from exclusions



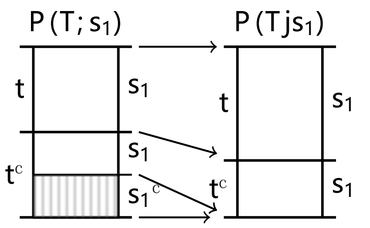
where $t^c = \{T \setminus t\}$ and $s_1{}^c = \{S_1 \setminus s_1\}$

Motivation

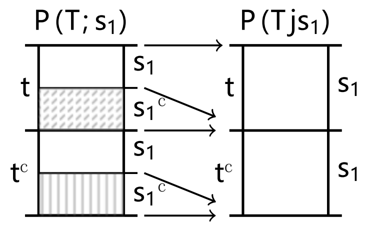


- The exclusions differ, but yet $i(t; s_1) = i(t; s_2) = 4/3$ bit
- Pointwise MI is not injective
- Same info $\Leftrightarrow$ same exclusions

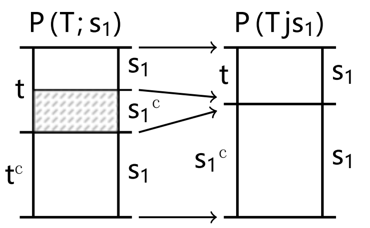
Two types of exclusions



Purely informative exclusion



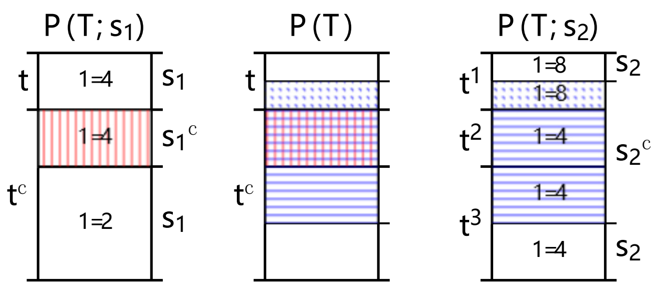
General case



Purely misinformative exclusions

- Idea: split the pointwise MI into two components
  $$i(s \rightarrow t) = i^{+}(s \rightarrow t) - i^{-}(s \rightarrow t)$$
- In Finn et al. (2017) we proved that
  (Specificity) $i^{+}(s_1 \rightarrow t) = h(s_1)$
  (Ambiguity) $i^{-}(s_1 \rightarrow t) = h(s_1|t)$

Specificity and ambiguity decomposition



$$i(s_1 \rightarrow t) = i(s_2 \rightarrow t) = \log \tfrac{4}{3} \text{ bit}$$

$i_{+}(s_1 \rightarrow t) = \log \frac{4}{3}$ bit; $\quad i_{-}(s_1 \rightarrow t) = 0$ bit

$i_{+}(s_2 \rightarrow t) = \log \frac{8}{3}$ bit; $\quad i_{-}(s_2 \rightarrow t) = 1$ bit

PPID using Specificity and Ambiguity

Axioms (PPID using Specificity and Ambiguity)

(1) Symmetry: $r^{\pm}(t : s_1; \ldots; s_n)$ is invariant under permutations of the s_i's
(2) Monotonicity: $r^{\pm}(t : s_1; \ldots; s_n) \leq r^{\pm}(t : s_1; \ldots; s_{n-1})$
(3) Self-redundancy: $r^{\pm}(t : s_i) = i^{\pm}(t; s_i)$

- Yields two redundancy lattices: the specificity and ambiguity lattices
- No longer have the non-negativity problem
- Still need a measure of redundant information on each lattice

PPID using Specificity and Ambiguity

Additional Axiom (PPID using Specificity and Ambiguity)

(4) Pointwise event space: $r^{\pm}(t : s_1; \ldots; s_n)$ depend only on the size of informative and misinformative exclusions

- Akin to Assumption ([illegible]) of Bertschinger et al. (2014) → cannot infer complementary info (synergy) without full joint distribution.
- Leads us to define the redundant specificity and redundant ambiguity

$$r^{+}_{\min}(s_1; \ldots; s_k \to t) = \min_{s_j} h(s_j) \qquad r^{-}_{\min}(s_1; \ldots; s_k \to t) = \min_{s_j} h(s_j | t)$$

Example: PwUnq

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
¼	0	1	1	1	1	2	1	2	1	1	0	1	0	1	0	0	0
¼	1	0	1	2	1	1	1	2	1	1	1	0	0	1	0	0	0
¼	0	2	2	1	1	2	1	2	1	1	0	1	0	1	0	0	0
¼	2	0	2	2	1	1	1	2	1	1	1	0	0	1	0	0	0
Expected values				3/2	1	3/2	1	2	1	1	½	½	0	1	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1; S_2) = 1 - 1 = 0 \text{ bit} \qquad U(T : S_1 \backslash S_2) = 1/2 - 0 = 1/2 \text{ bit}$$
$$C(T : S_1; S_2) = 0 - 0 = 0 \text{ bit} \qquad U(T : S_2 \backslash S_1) = 1/2 - 0 = 1/2 \text{ bit}$$

- Matches the PPID suggested earlier

Example: XOR

p	s_1	s_2	t	i_1^+	i_1^-	i_2^+	i_2^-	i_{12}^+	i_{12}^-	r^+	u_1^+	u_2^+	c^+	r^-	u_1^-	u_2^-	c^-
1/4	0	0	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	0	1	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	0	1	1	1	1	1	2	1	1	0	0	1	1	0	0	0
1/4	1	1	0	1	1	1	1	2	1	1	0	0	1	1	0	0	0
Expected values				1	1	1	1	2	1	1	0	0	1	1	0	0	0

- Recombining the average specificities and average ambiguities yields the PID

$$R(T : S_1; S_2) = 1 - 1 = 0 \text{ bit} \qquad U(T : S_1 \setminus S_2) = 0 - 0 = 0 \text{ bit}$$
$$C(T : S_1; S_2) = 1 - 0 = 1 \text{ bit} \qquad U(T : S_2 \setminus S_1) = 0 - 0 = 0 \text{ bit}$$

- Identifies redundancy due to shared knowledge from Bertschinger et al. (2014)

Further properties

- Can be interpreted in terms of Kelly gambling (e.g. difference in unique information is difference in winnings of two players)
- Is generalisable beyond two sources
- Has a target chain rule (in net):

$$r_{\min}^{-}\left(s_1; s_2 \to t_1; t_2\right) = r_{\min}^{-}\left(s_1; s_2 \to t_1\right) + r_{\min}^{-}\left(s_1; s_2 \to t_2 | t_1\right)$$

- Provides consistent decomposition of two-bit copy $I(\{S_1; S_2\}; S_1; S_2)$ via target chain rule
- Many echoes of intuition behind previous measures, e.g. (Bertschinger et al., 2014; Ince, 2017; Williams and Beer, 2010), but is fully pointwise and component-wise.

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