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An Innovative Circular Ring Method for Measuring Young's Modulus of Thin Flexible Multi-layered Materials †

Atsumi OHTSUKI [1,*]

[1] Department of Mechanical Engineering, Meijo University, Nagoya 468-8502, Japan
* E-mail: ohtsuki@ccmfs.meijo-u.ac.jp; Tel.: +81-52-832-1151
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Abstract: An innovative mechanical testing method (*Compressive Circular Ring Method*) is provided for measuring Young's modulus of each layer in a flexible multi-layered material. The method is based on a nonlinear large deformation theory. By just measuring the vertical displacement or the horizontal displacement of the ring, Young's modulus of each layer can be easily obtained for various thin multi-layered materials. Measurements were carried out on an electrodeposited two-layered wire. The results confirm that the new method is suitable for flexible multi-layered thin wires. In the meantime, the new method can be applied widely to measure Young's modulus of thin layers formed by PVD, CVD, Coating, Paint, Cladding, Lamination, and others.

Keywords: Young's Modulus; Multi-layered material; Circular ring; Large deformation

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1. Introduction

Young's modulus of multi-layered materials is very important to predict large deformation in both analytical and technological interests. A new testing method (*Circular Ring Method*) is based on a nonlinear theory. This paper deals with the compressive technique. Exact analytical solutions are obtained in terms of elliptic integrals. In order to assess the applicability of the proposed method, several experiments were carried out using a two-layered material (Cu: an electrodeposited material + SWPA: a spring steel material). As a result, the new method was found to be suitable for flexible multi-layered materials. Besides the *Circular Ring Method* studied here, the *Axial Compression Method* [1], the *Own-weight Cantilever Method* [2,3] for a flexible multi-layered material have already been developed and reported, based on the nonlinear large deformation theory.

2. Fundamental theory

A typical load-deformation shape is given in **Fig.1** for a circular ring (the initial radius: R_0, the whole length of the circular ring: $4L=2\pi R_0$) subjected to opposite compressive forces at two points. As an example, **Fig.2** shows the cross-section of a two-layered material.

The analysis is carried out for only the 1/4 part (Region AB, arc length L). The horizontal displacement is denoted by x, the vertical displacement by y, and θ is the deflection angle. Furthermore, the arc length is denoted by s, the radius of curvature by R and the bending moment by M. The relationship among R, M, s, x, y and θ are given by:

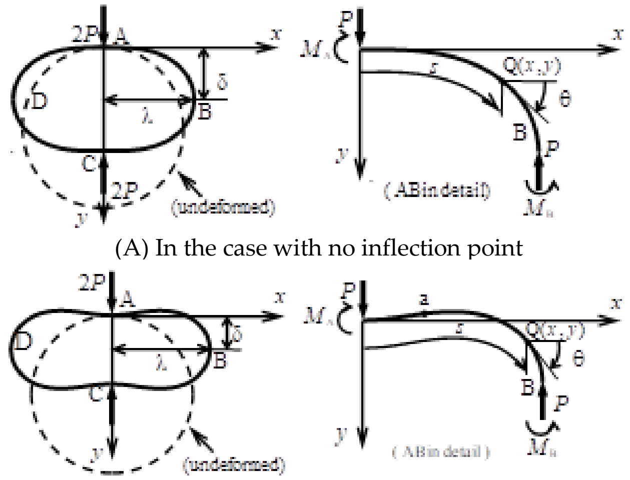
$$\left.\begin{aligned} 1/R = -d\theta/ds = M\Big/\sum_{i=1}^{n}(E_i I_i) + 1/R_0 \\ dy = ds\cdot\sin\theta,\, dx = ds\cdot\cos\theta \end{aligned}\right\} \tag{1}$$

where $E_i I_i$ = the flexural rigidity of each layer.

The bending moment applied at an arbitrary position Q(x, y) is expressed as

$$M = -P\cdot x + M_{\mathrm{A}} \tag{2}$$

Introducing the following non-dimensional variables,



(A) In the case with no inflection point

(B) In the case with inflection point a

Figure 1. The co-ordinate system for a flexible multi-layered circular ring subjected to opposite compressive forces.

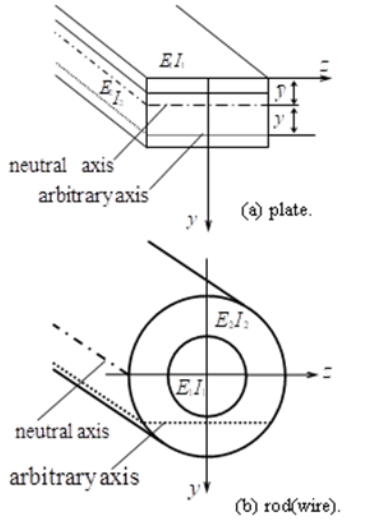


Figure 2. Illustration of cross-section of two-layered material (as an example).

$$\xi = x/L,\ \eta = y/L,\ \zeta = s/L,\ \ \rho = R/L,\ \ \gamma = PL^2 \Big/ \sum_{i=1}^{n}(E_i I_i),\ \alpha = M_A L \Big/ \sum_{i=1}^{n}(E_i I_i) \tag{3}$$

Considering the boundary condition, $\left(d\theta/d\zeta\right)\Big|_{\theta=\theta_A=0} = -\alpha - 1/\rho_0$ at the point A, the basic equation is derived from Eqs.(1), (2) and (3) in the form of :

$$d\theta/d\zeta = \pm\sqrt{2\gamma\sin\theta + \left(\alpha + 1/\rho_0\right)^2} \tag{4}$$

This nonlinear differential equation (4) is the basic equation that determines large deformation behaviors of a compressive ring.

$$\left.\begin{aligned} & k = \sqrt{\left\{2\gamma + \left(\alpha + 1/\rho_0\right)^2 / (4\gamma)\right\}} \\ & 1 - \sin\theta = 2k^2 \sin^2\phi \qquad \left(0 \le \phi \le \pi/2\right) \end{aligned}\right\} \tag{5}$$

2.1. In the case with no inflection point [see Fig.1(A)]

2.1.1. Coverage $0 \le k \le 1$ of the variable k in equation (5)

Considering the boundary conditions $\zeta_{max}(= s_{max}/L) = 1$, $\eta_{max} = \delta/L$ and $\xi_{max} = \lambda/L$, the maximum non-dimensional arc length ζ_{AB}, the maximum non-dimensional vertical displacement η_{AB} and the maximum non-dimensional horizontal displacement ξ_{AB} are obtained as follows.

$$\zeta_{AB} = 1 = F\left(k, \phi_A\right) / \sqrt{\gamma} \tag{6}$$

$$\eta_{AB} = \delta/L = \left\{2E\left(k, \phi_A\right) - F\left(k, \phi_A\right)\right\} / \sqrt{\gamma} \tag{7}$$

$$\xi_{AB} = \lambda/L = 2k \cdot \left(1 - \cos\phi_A\right) / \sqrt{\gamma} \tag{8}$$

Similarly, the non-dimensional load γ is

$$\gamma = PL^2/(EI) = \left\{F\left(k, \phi_A\right)\right\}^2 \tag{9}$$

where $\phi_A = \mathrm{Sin}^{-1}\left[\sqrt{\left\{1/\left(2k^2\right)\right\}}\right]$

2.1.2. Coverage $k \ge 1$ of the variable k in equation (5)

By transforming the variables $k\sin\phi = \sin Z$ the maximum non-dimensional arc length ζ_{AB}, the maximum non-dimensional vertical displacement η_{AB} and the maximum non-dimensional horizontal displacement ξ_{AB} are obtained as follows.

$$\zeta_{AB} = 1 = F(1/k\,, \pi/4)/(k\sqrt{\gamma}) \quad (10)$$

$$\eta_{AB} = \delta/L = \{2k \cdot E(1/k\,, \pi/4) - (2k - 1/k) \cdot F(1/k\,, \pi/4)\}/\sqrt{\gamma} \quad (11)$$

$$\xi_{AB} = \lambda/L = 2k \cdot (1 - \cos\phi_A)/\sqrt{\gamma} \quad (12)$$

where $\phi_A = \mathrm{Sin}^{-1}\left[\sqrt{\{1/(2k^2)\}}\right]$

Similarly, the non-dimensional load γ is

$$\gamma = PL^2/(EI) = \{F(1/k\,, \pi/4)/k\}^2\,, \quad (13)$$

2.2. In the case with inflection point [see Fig.1(B)]

In this case, a measuring theory can be derived under the coverage of the variable *k*, $0 \le k \le 1$. Details of the analytical theory will be omitted here.

Equations (6)–(13) are fundamental formulas to obtain Young's modulus of each layer, based on the nonlinear large deformation theory. The functions $F(k,\phi)$, $E(k,\phi)$ appeared in Eqs.(6)-(13) are Legendre–Jacobi's elliptic integrals of the first and second kinds, respectively.

The following formula based on Eq.(3) is useful in calculating each Young's modulus E_i.

$$\sum_{i=1}^{n}(E_i I_i) = PL^2/\gamma \quad (14)$$

where I_i is the second moment of area.

When calculating Young's modulus E_i using Eq.(14), it is not necessary to determine the neutral axis for multi-layered rods/wires because the cross section is symmetrical at any time with respect to the neutral axis. The second moment of area I_i of each cross section for multi-layered rods/wires (diameter d_i) with respect to the neutral axis is shown as

$$I_i = \pi\left(d_i^{\,4} - d_{i-1}^{\,4}\right)/64 \qquad (d_0 = 0) \quad (15)$$

On the other hand, in case of multi-layered plates it is necessary to determine the neutral axis of materials. The second moment of area I_i of each cross section (thickness h_i, width b: common to all) with respect to the neutral axis is shown as

$$I_i = bh_i^3/12 + bh_i\left(\bar{y} - h_i/2 - \sum_{k=1}^{i} h_{k-1}\right)^2 \qquad (h_0 = 0,\, i \le n) \quad (16)$$

The distance $\bar{y}$ to the neutral axis (see Fig.2(a)) is obtained as follows.

$$\bar{y} = \sum_{i=1}^{n} E_i (S_i)_z \Big/ \sum_{i=1}^{n}(E_i A_i) \quad (17)$$

The first moment of area $(S_i)_z$ of each cross section (A_i: the cross-sectional area) with respect to *z* axis is expressed as

$$(S_i)_z = bh_i^2/2 + bh_i\left(\sum_{k=1}^{i} h_{k-1}\right) \quad (18)$$

One quantity γ (: the non-dimensional load) is required to calculate Young's modulus E_i from Eq. (14). The value of γ is obtained from a chart (: Nomograph) of γ-δ relation (δ: the vertical displacement) [Method 1] or γ-λ relation (λ: the horizontal displacement) [Method 2].

3. Techniques of new measuring method (Compressive Circular Ring Method)

In this paper, two methods are introduced in order to measure Young's modulus. The γ-δ and γ -λ relations are presented in **Figs.3** and **4,** respectively. These charts are computed previously by using Eqs.(9). (13). Here, the usage of the chart is recommend by the author. As a point to note, for example, a two-step procedure should be done in a measuring experiment, when Young's modulus of each layer in a two-layered material is all unknown (Note that a multi-layered material with number of layers *n* requires a *n*- step procedure). In other words, it is possible to reduce a frequency of step in proportion to the number, if the number of layers with known Young's modulus is proven.

3.1. Method 1: (Measurement of δ only)

The usage of this method is shown below in a two-layered material. Each Young's modulus E_i is obtained for a SWPA thin wire (: first layer) with 1/4 part length: L_1=125.0[mm](4L_1(500 [mm]): whole length of the ring), diameter: d_1=0.38[mm] and a Cu electrodeposited layer (: second layer) with length: L_2 (=L_1) =125.0[mm], thickness: $(d_2–d_1)/2$=0.011[mm] (d_2: 0.402 [mm]).

A chart (: Nomograph) is given in **Fig.3,** illustrating the relationship of γ and δ/L. Using this chart, each Young's modulus E_i in a multi-layered material can be calculated from the relational expression given in Eq.(14).

3.1.1. **First step procedure (As a two-layered specimen)**

Under the condition of P=39.24[mN], δ=64.7[mm] (i.e., δ/L=0.5176) is measured for a double layer and then the value of γ is taken from **Fig.3** (γ=2.470). Therefore, using Eq.(14), the combined flexural rigidity (I_1=1.023×10^{-15}[m^4]: SWPA, I_2=2.584×10^{-16}[m^4]: Cu) is derived as follows

$$E_1I_1 + E_2I_2 = PL_1^{\,2}/\gamma = 0.03924\times 0.125^2/2.470 = 2.480\times 10^{-4} \tag{19}$$

3.1.2. **Second step procedure (As a single-layered specimen)**

Similarly, δ is measured for a single layer after removing a second layer (Cu). In the condition of P=34.34[mN] for a SWPA single layer with length: L_1=125.5[mm], diameter: d_1=0.38[mm], δ=64.4[mm] (i.e., δ/L=0.5148) is measured and γ is taken newly from **Fig.3*** (γ=2.51) [*: Drawing is omitted here.]. Therefore, the flexural rigidity (I_1=1.023×10^{-15}[m^4]: SWPA) can be rewritten as follows from Eq.(14) follows.

$$E_1I_1 = PL_1^{\,2}/\gamma = 0.03434\times 0.125^2/2.51 = 2.135\times 10^{-4} \tag{20}$$

Using the simultaneous equations (19) and (20), Young's modulus E_1, E_2 of each layer is

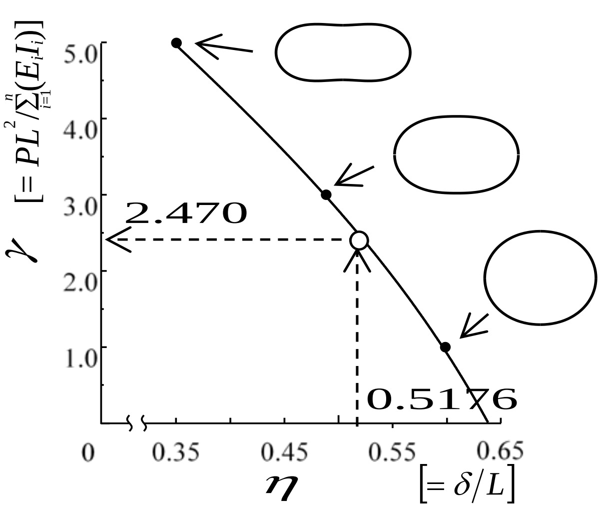


Figure 3. Non-dimensional chart for the parameter γ when the vertical displacement δ is given.

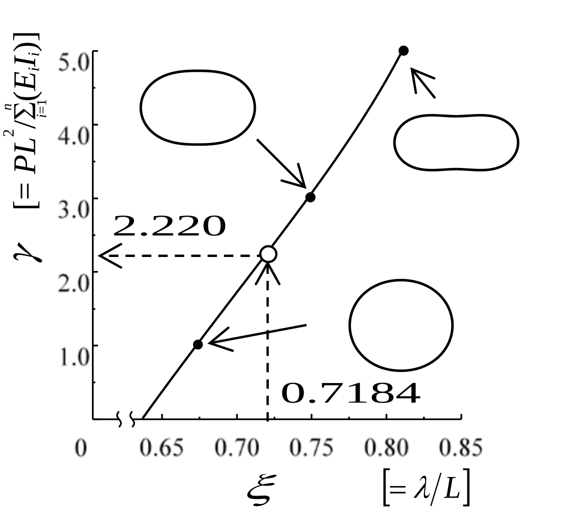


Figure 4. Non-dimensional chart for the parameter γ when the horizontal displacement λ is given.

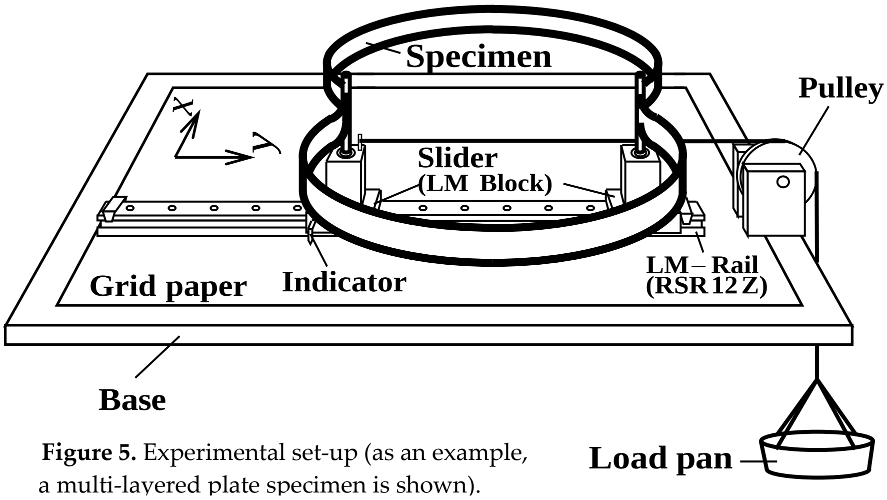


Figure 5. Experimental set-up (as an example, a multi-layered plate specimen is shown).

calculated as E_1=209.3 [GPa] for a SWPA layer and E_2=129.9 [GPa] for a Cu layer.

3.2. Method 2: (Measurement of λ only)

A similar chart (: Nomograph) is given in **Fig.4**, illustrating the relationship of γ and λ/L. Using this chart, each Young's modulus E_i in a multi-layered material can be calculated from Eq.(14). As an example, Young's modulus E_i of each layer is obtained for a SWPA thin wire (: first layer) + a Cu thin layer (: second layer) mentioned above (see the tertiary section 3.1).

3.2.1. **First step procedure (As a two-layered specimen)**

Under the condition of P=34.34[mN], λ =89.5[mm] (i.e., λ/L=0.7184) are measured for a double layer and then the value of γ is taken from **Fig.4** (γ =2.220). Therefore, from Eq.(14) the combined flexural rigidity (I_1=1.023×10^{-15}[m^4]: SWPA, I_2=2.584×10^{-16}[m^4]: Cu) can be written as follows

$$E_1 I_1 + E_2 I_2 = P{L_1}^2/\gamma = 0.03434 \times 0.125^2/2.220 = 2.414 \times 10^{-4} \quad (21)$$

3.2.2. **Second step procedure (As a single-layered specimen)**

Similarly, λ is measured for a single layer after removing a second layer (Cu). In the condition of P=29.43[mN] for a SWPA single layer, λ =89.5 [mm] (i.e., λ/L=0.716) is measured and then γ is taken newly from **Fig.4*** (γ=2.15) [*: Drawing is omitted here.]. Therefore, the flexural rigidity (I_1=1.023×10^{-15}[m^4]: SWPA) can be rewritten as follows from Eq.(14).

$$E_1 I_1 = P{L_1}^2/\gamma = 0.02943 \times 0.125^2/2.15 = 2.136 \times 10^{-4} \quad (22)$$

From the simultaneous equations (21) and (22), Young's modulus E_1, E_2 of each layer is calculated as E_1=209.5 [GPa] for a SWPA layer and E_2=104.57 [GPa] for a Cu layer.

4. Experimental investigation

In order to assess the applicability of the Compressive Circular Ring Method, several large deformation experiments were carried out using a two-layered wire [Cu (Copper) layer: an electrodeposited material (0.011mm thick, 500mm long) + SWPA layer: a spring steel wire (0.38mm diameter, 500mm long)]. The experimental set-up is shown in **Fig.5** (which shows a thin multi-layered plate, for example). Since Young's modulus of each layer in the two-layered material is unknown, the measuring experiments were carried out by adopting the two-step procedure. In every step of the procedures, a vertical displacement δ and a horizontal displacement λ are measured for several compressive loads P by using a grid paper with 1mm spacing.

Young's moduli of Cu and SWPA obtained by applying Method 1 and Method 2 are shown in **Figs. 6 and 7**, respectively. Here, the influence of a load (P) upon Young's modulus (E) was examined.

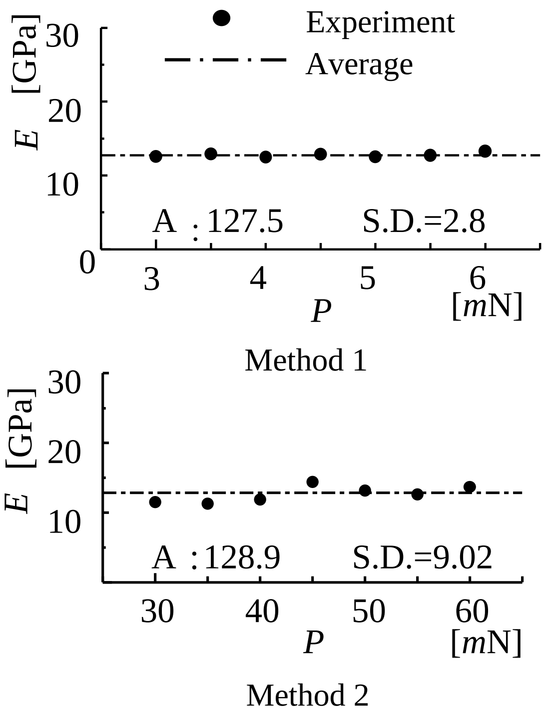


Figure 6. Comparison of Young's moduli of an electrodeposited material (Cu: E_2) between the two measuring methods for various values of the load *P*. (Note: E_1 of SWPA is known previously.)

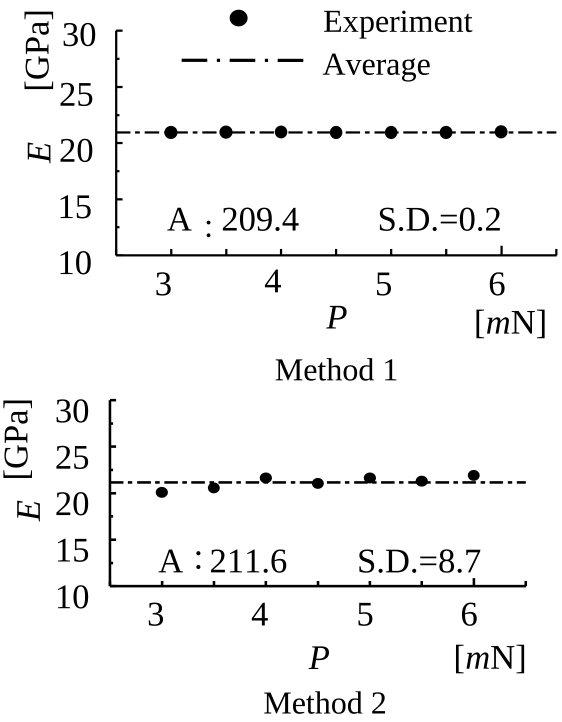


Figure 7. Comparison of Young's moduli of a spring steel wire (SWPA: E_1) between the two measuring methods for various values of the load *P*. (Note: E_2 of Cu is known previously.)

The figures were described under a two-layered condition

In a Cu layer (see **Fig.6**), the measured values of Methods 1 and 2 remain nearly constant for a compressive load and the standard deviation (S.D.) is very small although every method has a little scattered values. As a whole, the mean Young's moduli (shown as Av.: Average) determined by the two methods are reasonably in good agreement with each other. On the other hand, Trends similar to that of **Fig.6** is observed for Young's moduli of a SWPA layer (see **Fig.7**). The mean values obtained by the two methods agree well.

5. Conclusions

The "Compressive Circular Ring Method" is proposed as a new and simpler material testing method for measuring Young's modulus of each layer in a flexible multi-layered material.

From the results of theoretical and experimental analyses, the new method is effective for measuring Young's modulus of each layer in a flexible multi-layered material. Furthermore, the proposed new method is applicable widely to Young's modulus measurement in a thin layer formed, for example, by PVD (Physical Vapor Deposition), CVD(Chemical Vapor Deposition), Coating (Graphite, Metal Oxide), Paint(Lacquer) , etc.

Author Contributions: Ohtsuki.A. performed the theoretical analysis, conceived and designed the experiments, and wrote the paper.

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